Max. Marks : 90

Time : 120 Minutes

HST – 01 A

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INDIRA Award Winner

Topics : Chapter 02 – 08 of NCERT Textbooks (Based on 2015 Exams.) M.+91 9650350480

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NOTE : (i) Internal Choices has been increased only to include all the sums asked in recent CBSE Board Exams 2015 of various regions viz. Delhi, All India (various regions) and Foreign.
(ii) Although Section C is usually asked in **Six** marks, here it is of **Five** marks.

All the Best!

SECTION – A ***(Each question carry One mark.)***

Q01. If X is an order n matrix such that $\text{adj}(\lambda A) = k.\text{adj}A$, then find the value of k.

Q02. What is the principal domain of $\text{cosec}^{-1}x$?

Q03. Is $x = -\frac{1}{2}$ a solution of the equation $\sin^{-1}(1-x) - 2\sin^{-1}x = \frac{\pi}{2}$? Justify your answer.

Q04. Draw a free-hand graph of $\sin^{-1}x$.

SECTION – B ***(Each question carry Four marks.)***

Q05. If A and B are two matrices such that $AB = B$ and $BA = A$ then, show that $A^2 + B^2 = A + B$.

OR If $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}$ and $(A+B)^2 = A^2 + B^2$, then find the values of a and b.

Q06. If $A = \begin{pmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{pmatrix}$ find $A^2 - 5A + 4I$ and hence find a matrix X such that $A^2 - 5A + 4I + X = O$.

Q07. To raise money for an orphanage, students of three schools P, Q and R organized an exhibition in their area, where they sold paper bags, used books and files made by them using recycled paper at the rate of ₹15, ₹12 and ₹10 per unit. School P sold 20 paper bags, 15 books and 13 files. School Q sold 16 paper bags, 15 books and 15 files and, school R sold 25 paper bags, 22 books and 15 files. Using matrices, find the total amount raised by each school.
By such exhibition, which values are inculcated in the students?

Q08. Using properties of determinants, prove the following :

$$\begin{vmatrix} 1 & a & a^2 \\ a^2 & 1 & a \\ a & a^2 & 1 \end{vmatrix} = (1-a^3)^2.$$

OR Using properties of determinants, prove that $\begin{vmatrix} a^3 & 2 & a \\ b^3 & 2 & b \\ c^3 & 2 & c \end{vmatrix} = 2(a-b)(b-c)(c-a)(a+b+c)$.

Q09. Evaluate : $\cos\left(\frac{1}{2}\sin^{-1}\frac{3}{5}\right)$.

Q10. Prove that : $\cot^{-1}x + \cot^{-1}y = \cot^{-1}\left(\frac{xy-1}{x+y}\right)$.

OR If $\sin[\cot^{-1}(x+1)] = \cos(\tan^{-1}x)$, then find x.

Q11. If $2\tan^{-1}(\cos\theta) = \tan^{-1}(2\,\text{cosec}\,\theta), (\theta \neq 0)$, then find the value of θ.

Q12. If $\tan^{-1}\left(\frac{1}{1+1.2}\right) + \tan^{-1}\left(\frac{1}{1+2.3}\right) + ... + \tan^{-1}\left(\frac{1}{1+n.(n+1)}\right) = \tan^{-1}\theta$, then find the value of θ.

OR If $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \pi$, then prove that $x\sqrt{1-x^2} + y\sqrt{1-y^2} + z\sqrt{1-z^2} = 2xyz$.

Q13. If $x = a\cos\theta + b\sin\theta$ and, $y = a\sin\theta - b\cos\theta$, show that $y^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$.

Q14. Discuss the continuity and differentiability of the function $f(x) = |x| + |x-1|$ in the interval $(-1, 2)$.

OR Show that the function $f(x) = |x-1| + |x+1|$, for all $x \in R$, is not differentiable at the points $x = -1$ and $x = 1$.

Q15. If $f(x) = \sqrt{x^2+1}$; $g(x) = \frac{x+1}{x^2+1}$ and $h(x) = 2x - 3$, then find $f'[h'\{g'(x)\}]$.

Q16. If $x = a(\cos 2t + 2t\sin 2t)$ and $y = a(\sin 2t - 2t\cos 2t)$, then find $\frac{d^2y}{dx^2}$.

OR If the derivative of $\tan^{-1}(a+bx)$ takes the value 1 at $x = 0$, prove that $1 + a^2 = b$.

Q17. Find the local maxima and local minima, of the function $f(x) = \sin x - \cos x,\ 0 < x < 2\pi$. Also find the local maximum and local minimum values.

OR The side of an equilateral triangle is increasing at the rate of 2cm/s. At what rate is its area increasing when the side of the triangle is 20cm?

Q18. Integrate $\frac{x^2 - 3x + 1}{\sqrt{1-x^2}}$ w.r.t. x. **OR** Evaluate : $\int (3-2x)\sqrt{2+x-x^2}\,dx$.

Q19. Find the anti-derivative of $\frac{\sin 2x - \sin 2k}{\sin x - \sin k + \cos x - \cos k}$.

Q20. Find : $\int \frac{\log x\ dx}{(x+1)^2}$.

Q21. Evaluate : $\int \frac{x^2+x+1}{(x^2+1)(x+2)}dx$. **OR** Evaluate : $\int \frac{x^3}{(x-1)(x^2+1)}dx$.

Q22. Evaluate : $\int_{-\pi/2}^{\pi/2} \frac{\cos x}{1+e^x}dx$.

Q23. Evaluate : $\int_0^{\pi/4} \left(\frac{\sin x + \cos x}{3 + \sin 2x}\right)dx$. **OR** Evaluate : $\int_0^{\pi/2} \frac{\cos^2 x dx}{1 + 3\sin^2 x}$.

SECTION – C ***(Each question carry Five marks.)***

Q24. Find the area enclosed between the curves $y = x^2$ and $y^2 = x$.

Q25. Find the value of p for when the curves $x^2 = 9p(9-y)$ and $x^2 = p(y+1)$ cut each other at right angles.

OR Find the minimum value of $(ax + by)$, where $xy = c^2$.

HALF SYLLABUS TEST – 01 Set A [Complete Solutions]

SECTION – A

Q01. $k=\lambda^{n-1}$, where n is the order of matrix .

Q02. $R-(-1,\,1)$

Q03. Consider LHS and replace $x=-\frac{1}{2}$: $\sin^{-1}\left(1-\left(-\frac{1}{2}\right)\right)-2\sin^{-1}\left(-\frac{1}{2}\right)=\sin^{-1}\left(\frac{3}{2}\right)-2\left(-\frac{\pi}{6}\right)\neq\frac{\pi}{2}$ of RHS.

So $x=-\frac{1}{2}$ isn't a solution of the given equation.

Also, note that $\sin^{-1}\left(\frac{3}{2}\right)$ can't be possible as domain of $\sin^{-1}x$ is $x\in[-1,\,1]$.

Q04. See NCERT Page 35 Fig. 2.1 (ii)

SECTION – B

Q05. Given $AB=B$ $\Rightarrow ABB^{-1}=BB^{-1}$ $\Rightarrow AI=I$ i.e., $A=I$.
Similarly we have $BA=A$ $\Rightarrow B=I$.
Consider LHS: $A^2+B^2=I^2+I^2=I.I+I.I=I+I=A+B=RHS$

OR If $A=\begin{bmatrix}1 & -1\\ 2 & -1\end{bmatrix}$ and $B=\begin{bmatrix}a & 1\\ b & -1\end{bmatrix}$ and $(A+B)^2=A^2+B^2$, then find the values of a and b.

We've $(A+B)^2=A^2+B^2$ $\Rightarrow (A+B)(A+B)=A^2+B^2$ $\Rightarrow A.A+A.B+B.A+B.B=A.A+B.B$

$\Rightarrow A.B=-B.A$ So, $\begin{bmatrix}1 & -1\\ 2 & -1\end{bmatrix}\begin{bmatrix}a & 1\\ b & -1\end{bmatrix}=-\begin{bmatrix}a & 1\\ b & -1\end{bmatrix}\begin{bmatrix}1 & -1\\ 2 & -1\end{bmatrix}$

$\Rightarrow\begin{bmatrix}a-b & 2\\ 2a-b & 3\end{bmatrix}=\begin{bmatrix}-a-2 & a+1\\ 2-b & b-1\end{bmatrix}$ By equality of matrices, we get :

$a-b=-a-2,\ 2=a+1, 2a-b=2-b, 3=b-1$

On solving these equations, we get : $a=1,\ b=4$.

Q06. We have $A=\begin{pmatrix}2 & 0 & 1\\ 2 & 1 & 3\\ 1 & -1 & 0\end{pmatrix}$ $\therefore A^2=A.A=\begin{pmatrix}2 & 0 & 1\\ 2 & 1 & 3\\ 1 & -1 & 0\end{pmatrix}\begin{pmatrix}2 & 0 & 1\\ 2 & 1 & 3\\ 1 & -1 & 0\end{pmatrix}=\begin{pmatrix}5 & -1 & 2\\ 9 & -2 & 5\\ 0 & -1 & -2\end{pmatrix}$

So, $A^2-5A+4I=\begin{pmatrix}5 & -1 & 2\\ 9 & -2 & 5\\ 0 & -1 & -2\end{pmatrix}-5\begin{pmatrix}2 & 0 & 1\\ 2 & 1 & 3\\ 1 & -1 & 0\end{pmatrix}+4\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$

$\Rightarrow A^2-5A+4I=\begin{pmatrix}5 & -1 & 2\\ 9 & -2 & 5\\ 0 & -1 & -2\end{pmatrix}-\begin{pmatrix}10 & 0 & 5\\ 10 & 5 & 15\\ 5 & -5 & 0\end{pmatrix}+\begin{pmatrix}4 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 4\end{pmatrix}$

$\therefore A^2-5A+4I=\begin{pmatrix}-1 & -1 & -3\\ -1 & -3 & -10\\ -5 & 4 & 2\end{pmatrix}$.

Also $A^2-5A+4I+X=O$ $\Rightarrow\begin{pmatrix}-1 & -1 & -3\\ -1 & -3 & -10\\ -5 & 4 & 2\end{pmatrix}+X=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$ $\therefore X=\begin{pmatrix}1 & 1 & 3\\ 1 & 3 & 10\\ 5 & -4 & -2\end{pmatrix}$.

Q07. Let the amounts raised by schools P, Q and R be x, y and z (in ₹) respectively.

Paper bags Used Books Files Cost

So, Amounts by school P → ; Amounts by school Q → ; Amounts by school R →

$$\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}20 & 15 & 13\\ 16 & 15 & 15\\ 25 & 22 & 15\end{bmatrix}\begin{bmatrix}15\\ 12\\ 10\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 300+180+130 \\ 240+180+150 \\ 375+264+150 \end{bmatrix} = \begin{bmatrix} 610 \\ 570 \\ 789 \end{bmatrix}$$

By equality of matrices, we get : $x = 610,\ y = 570,\ z = 789$.

So the amounts raised by school P is ₹610, by school Q is ₹570 and by school R is ₹789.

By such exhibition, the values of helpfulness and sympathy are inculcated in the students.

Q08. LHS : Let $\Delta = \begin{vmatrix} 1 & a & a^2 \\ a^2 & 1 & a \\ a & a^2 & 1 \end{vmatrix}$ By $C_1 \to C_1 - aC_3$

$$\Rightarrow \quad = \begin{vmatrix} 1-a^3 & a & a^2 \\ 0 & 1 & a \\ 0 & a^2 & 1 \end{vmatrix} \qquad \text{Taking } \left(1-a^3\right) \text{ common from } C_1$$

$$\Rightarrow \quad = \left(1-a^3\right)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & a \\ 0 & a^2 & 1 \end{vmatrix} \qquad \text{By } R_2 \to R_2 - aR_3$$

$$\Rightarrow \quad = \left(1-a^3\right)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1-a^3 & 0 \\ 0 & a^2 & 1 \end{vmatrix} \qquad \text{Taking } \left(1-a^3\right) \text{ common from } R_2$$

$$\Rightarrow \quad = \left(1-a^3\right)^2\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & 0 \\ 0 & a^2 & 1 \end{vmatrix} \qquad \text{Expanding along } C_1$$

$$\Rightarrow \quad = \left(1-a^3\right)^2\left\{1(1-0)-0+0\right\} \qquad \therefore \Delta = \left(1-a^3\right)^2 = \text{RHS}$$

OR LHS : Let $\Delta = \begin{vmatrix} a^3 & 2 & a \\ b^3 & 2 & b \\ c^3 & 2 & c \end{vmatrix}$ By $R_1 \to R_1 - R_2,\ R_2 \to R_2 - R_3$

$$\Rightarrow \Delta = \begin{vmatrix} a^3-b^3 & 0 & a-b \\ b^3-c^3 & 0 & b-c \\ c^3 & 2 & c \end{vmatrix} \qquad \text{Taking (a – b) \& (b – c) common from } R_1 \text{ and } R_2 \text{ respectively}$$

$$\Rightarrow \Delta = (a-b)(b-c)\begin{vmatrix} a^2+ab+b^2 & 0 & 1 \\ b^2+bc+c^2 & 0 & 1 \\ c^3 & 2 & c \end{vmatrix} \qquad \text{By } R_1 \to R_1 - R_2$$

$$\Rightarrow \Delta = (a-b)(b-c)\begin{vmatrix} (a-c)(a+c)+b(a-c) & 0 & 0 \\ b^2+bc+c^2 & 0 & 1 \\ c^3 & 2 & c \end{vmatrix} \qquad \text{Taking } (c-a) \text{ common from } R_1$$

$$\Rightarrow \Delta = (a-b)(b-c)(c-a)\begin{vmatrix} -a-b-c & 0 & 0 \\ b^2+bc+c^2 & 0 & 1 \\ c^3 & 2 & c \end{vmatrix} \qquad \text{Expanding along } R_1$$

$$\Rightarrow \Delta = (a-b)(b-c)(c-a)\left\{(-a-b-c)[0-2]-0+0\right\}$$

$$\therefore \Delta = 2(a-b)(b-c)(c-a)(a+b+c) = \text{RHS}.$$

Q09. Consider $y = \cos\left(\frac{1}{2}\sin^{-1}\frac{3}{5}\right)$ Let $\frac{1}{2}\sin^{-1}\frac{3}{5} = \theta \Rightarrow \sin 2\theta = \frac{3}{5} \Rightarrow \cos 2\theta = \frac{4}{5}$

$\therefore y=\cos\theta=\sqrt{\frac{1+\cos 2\theta}{2}}$ $\Rightarrow y=\sqrt{\frac{1+\frac{4}{5}}{2}}$ $\therefore y=\frac{3}{\sqrt{10}}$

Q10. Let $\cot^{-1}x=\alpha,\ \cot^{-1}y=\beta\ldots(i)$ $\Rightarrow x=\cot\alpha,\ y=\cot\beta$

$\because \cot(\alpha+\beta)=\frac{\cot\beta\cot\alpha-1}{\cot\beta+\cot\alpha}$ $\Rightarrow \cot(\alpha+\beta)=\frac{yx-1}{y+x}$ $\Rightarrow \alpha+\beta=\cot^{-1}\left(\frac{yx-1}{y+x}\right)$

By using (i), we get : $\cot^{-1}x+\cot^{-1}y=\cot^{-1}\left(\frac{xy-1}{x+y}\right)$.

OR We've $\sin[\cot^{-1}(x+1)]=\cos(\tan^{-1}x)$ $\Rightarrow \sin\sin^{-1}\left(\frac{1}{\sqrt{1+(x+1)^2}}\right)=\cos\cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$

$\Rightarrow \frac{1}{\sqrt{1+(x+1)^2}}=\frac{1}{\sqrt{1+x^2}}$ $\Rightarrow 2+x^2+2x=1+x^2$ $\therefore x=-\frac{1}{2}$.

Q11. We've $2\tan^{-1}(\cos\theta)=\tan^{-1}(2\operatorname{cosec}\theta)$ $\Rightarrow \tan^{-1}\left(\frac{2\cos\theta}{1-\cos^2\theta}\right)=\tan^{-1}(2\operatorname{cosec}\theta)$

$\Rightarrow \tan\tan^{-1}\left(\frac{2\cos\theta}{1-\cos^2\theta}\right)=\tan\tan^{-1}\left(\frac{2}{\sin\theta}\right)$ $\Rightarrow \frac{2\cos\theta}{\sin^2\theta}=\frac{2}{\sin\theta}$

$\Rightarrow \cos\theta\sin\theta-\sin^2\theta=0$ $\Rightarrow \sin\theta(\cos\theta-\sin\theta)=0$ $\Rightarrow \sin\theta=0$ or $\cos\theta=\sin\theta$

$\Rightarrow \theta=0$ or $\theta=\frac{\pi}{4}$. But $\theta\neq 0$, so $\theta=\frac{\pi}{4}$ is the only required solution.

Q12. We've $\tan^{-1}\left(\frac{1}{1+1.2}\right)+\tan^{-1}\left(\frac{1}{1+2.3}\right)+\ldots+\tan^{-1}\left(\frac{1}{1+n.(n+1)}\right)=\tan^{-1}\theta$

$\Rightarrow \tan^{-1}\left(\frac{2-1}{1+1.2}\right)+\tan^{-1}\left(\frac{3-2}{1+2.3}\right)+\ldots+\tan^{-1}\left(\frac{n-(n-1)}{1+(n-1).n}\right)+\tan^{-1}\left(\frac{(n+1)-n}{1+n.(n+1)}\right)=\tan^{-1}\theta$

$\Rightarrow \tan^{-1}2-\tan^{-1}1+\tan^{-1}3-\tan^{-1}2+\ldots+\tan^{-1}n-\tan^{-1}(n-1)+\tan^{-1}(n+1)-\tan^{-1}n=\tan^{-1}\theta$

$\Rightarrow \tan^{-1}(n+1)-\tan^{-1}1=\tan^{-1}\theta$ $\Rightarrow \tan^{-1}\left(\frac{(n+1)-1}{1+(n+1).1}\right)=\tan^{-1}\theta$

$\Rightarrow \tan^{-1}\left(\frac{n}{n+2}\right)=\tan^{-1}\theta$ $\therefore \theta=\frac{n}{n+2}$.

OR Let $\sin^{-1}x=\alpha,\ \sin^{-1}y=\beta,\ \sin^{-1}z=\gamma\ \therefore \alpha+\beta+\gamma=\pi\ldots(i)$

Consider LHS : $x\sqrt{1-x^2}+y\sqrt{1-y^2}+z\sqrt{1-z^2}=\sin\alpha\sqrt{1-\sin^2\alpha}+\sin\beta\sqrt{1-\sin^2\beta}+\sin\gamma\sqrt{1-\sin^2\gamma}$

$\Rightarrow \quad =\sin\alpha\cos\alpha+\sin\beta\cos\beta+\sin\gamma\cos\gamma$ $\Rightarrow \quad =\frac{1}{2}[2\sin\alpha\cos\alpha+2\sin\beta\cos\beta+2\sin\gamma\cos\gamma]$

$\Rightarrow \quad =\frac{1}{2}[\sin 2\alpha+\sin 2\beta+\sin 2\gamma]$ $\Rightarrow \quad =\frac{1}{2}[2\sin(\alpha+\beta)\cos(\alpha-\beta)+2\sin\gamma\cos\gamma]$

$\Rightarrow \quad =\sin(\pi-\gamma)\cos(\alpha-\beta)+\sin\gamma\cos\gamma$ $\Rightarrow \quad =\sin\gamma\cos(\alpha-\beta)+\sin\gamma\cos\gamma$

$\Rightarrow \quad =\sin\gamma[\cos(\alpha-\beta)+\cos\gamma]$ $\Rightarrow \quad =\sin\gamma\left[2\cos\frac{\alpha-\beta+\gamma}{2}\cos\frac{\alpha-\beta-\gamma}{2}\right]$

$\Rightarrow \quad =\sin\gamma\left[2\cos\frac{\pi-\beta-\beta}{2}\cos\frac{\alpha+\alpha-\pi}{2}\right]$ $\Rightarrow \quad =\sin\gamma\left[2\cos\left(\frac{\pi}{2}-\beta\right)\cos\left(\frac{\pi}{2}-\alpha\right)\right]$

$\Rightarrow \quad =\sin\gamma[2\sin\beta\sin\alpha]=2xyz=\text{RHS}$. [By using (i)

Q13. Given $x=a\cos\theta+b\sin\theta$ and, $y=a\sin\theta-b\cos\theta$

$\Rightarrow \frac{dx}{d\theta}=-a\sin\theta+b\cos\theta$ and, $\frac{dy}{d\theta}=a\cos\theta+b\sin\theta$

$\therefore \frac{dy}{dx}=\frac{dy}{d\theta}\times\frac{d\theta}{dx}=\frac{a\cos\theta+b\sin\theta}{b\cos\theta-a\sin\theta}=-\frac{x}{y}\ldots(i)$

$\Rightarrow y\dfrac{dy}{dx} = -x$ $\quad\Rightarrow y\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx}\times\dfrac{dy}{dx} = -1$ $\quad$ By (i), we get :

$y\dfrac{d^2y}{dx^2} + \left(-\dfrac{x}{y}\right)\times\dfrac{dy}{dx} + 1 = 0$ $\quad\therefore y^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + y = 0$.

Q14. Since the continuity and differentiability of modulus function is doubtful at the corner points. So well shall check continuity and differentiability at the critical points x = 0, 1 $\in(-1,2)$.

$$\therefore f(x) = |x| + |x-1| = \begin{cases} -x-(x-1) = -2x+1, \text{ if } x<0 \\ x-(x-1) = 1, \text{ if } 0\le x<1 \\ x+x-1 = 2x-1, \text{ if } x\ge 1 \end{cases}$$

Continuity at x = 0 : We have f (0) = 1.

LHL (at x = 0) : $\lim\limits_{x\to0^-}(-2x+1) = -2\times0+1 = 1$

RHL (at x = 0) : $\lim\limits_{x\to0^+} 1 = 1$

Since $\lim\limits_{x\to0} f(x) = f(0)$ so f (x) is continuous at x = 0.

Continuity at x = 1 : We have f (1) = 2 (1) – 1 = 1.

LHL (at x = 1) : $\lim\limits_{x\to1^-} 1 = 1$

RHL (at x = 1) : $\lim\limits_{x\to1^+} 2x-1 = 2\times1-1 = 1$

Since $\lim\limits_{x\to1} f(x) = f(1)$ so f (x) is continuous at x = 1.

Differentiability at x = 0 :

LHD (at x = 0) : $\lim\limits_{x\to0^-}\dfrac{|x|+|x-1|-1}{x-0} = \lim\limits_{x\to0^-}\dfrac{-2x+1-1}{x} = \lim\limits_{x\to0^-}\dfrac{-2x}{x} = -2$

RHD (at x = 0) : $\lim\limits_{x\to0^+}\dfrac{1-1}{x-0} = \lim\limits_{x\to0^+}\dfrac{0}{x} = \lim\limits_{x\to0^+} 0 = 0 \ne \text{LHD (at } x=0)$

$\therefore$ f(x) is not differentiable at x = 0.

Differentiability at x = 1 :

LHD (at x = 1) : $\lim\limits_{x\to1^-}\dfrac{1-1}{x-1} = \lim\limits_{x\to1^-}\dfrac{0}{x-1} = \lim\limits_{x\to1^-} 0 = 0$

RHD (at x = 1) : $\lim\limits_{x\to1^+}\dfrac{x+x-1-1}{x-1} = \lim\limits_{x\to1^+}\dfrac{2(x-1)}{x-1} = 2 \ne \text{LHD (at } x=1)$

$\therefore$ f(x) is not differentiable at x = 1.

Note there is one more method to solve this sum. To view, click on theOPGupta.com/

OR W've $f(x) = |x-1| + |x+1| = \begin{cases} -(x-1)-(x+1) = -2x, \text{ if } x<-1 \\ -(x-1)+(x+1) = 2, \text{ if } -1\le x<1 \\ x-1+x+1 = 2x, \text{ if } x\ge 1 \end{cases}$

$\therefore f(-1) = 2,\ f(1) = 2\times1 = 2$

Differentiability at x = –1 :

LHD (at x = –1) : $\lim\limits_{x\to-1^-}\dfrac{-2x-2}{x-(-1)} = \lim\limits_{x\to-1^-}\dfrac{-2(x+1)}{x+1} = \lim\limits_{x\to-1^-}(-2) = -2$

RHD (at x = –1) : $\lim\limits_{x\to-1^+}\dfrac{2-2}{x-(-1)} = \lim\limits_{x\to-1^+}\dfrac{0}{x+1} = \lim\limits_{x\to-1^+} 0 = 0 \ne \text{LHD (at } x=-1)$

$\therefore$ f(x) is not differentiable at x = –1.

Differentiability at x = 1 :

LHD (at x = 1) : $\lim\limits_{x\to1^-}\dfrac{2-2}{x-1} = \lim\limits_{x\to1^-}\dfrac{0}{x-1} = \lim\limits_{x\to1^-} 0 = 0$

RHD (at x = 1) : $\lim\limits_{x\to1^+}\dfrac{2x-2}{x-1} = \lim\limits_{x\to1^+}\dfrac{2(x-1)}{x-1} = 2 \ne \text{LHD (at } x=1)$

$\therefore$ f(x) is not differentiable at x = 1.

Q15. We have $f(x)=\sqrt{x^2+1}$; $g(x)=\dfrac{x+1}{x^2+1}$ and $h(x)=2x-3$

$$\Rightarrow f'(x)=\frac{2x}{2\sqrt{x^2+1}};\ g'(x)=\frac{(x^2+1)-(x+1)2x}{(x^2+1)^2} \text{ and } h'(x)=2$$

$$\Rightarrow f'(x)=\frac{x}{\sqrt{x^2+1}};\ g'(x)=\frac{1-2x-x^2}{(x^2+1)^2} \text{ and } h'(x)=2$$

Now $f'[h'\{g'(x)\}]=f'\left[h'\left\{\dfrac{1-2x-x^2}{(x^2+1)^2}\right\}\right]=f'[2]=\dfrac{2}{\sqrt{2^2+1}}=\dfrac{2}{\sqrt{5}}$.

Q16. Given $x=a(\cos 2t+2t\sin 2t)$ and $y=a(\sin 2t-2t\cos 2t)$

$\therefore \dfrac{dx}{dt}=a(-2\sin 2t+4t\cos 2t+2\sin 2t)=4at\cos 2t$ & $\dfrac{dy}{dt}=a(2\cos 2t+4t\sin 2t-2\cos 2t)=4at\sin 2t$

So, $\dfrac{dy}{dx}=\dfrac{dy}{dt}\times\dfrac{dt}{dx}=\dfrac{4at\sin 2t}{4at\cos 2t}=\tan 2t$ $\qquad\Rightarrow \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)=\dfrac{d^2y}{dx^2}=2\sec^2 2t\times\dfrac{dt}{dx}$

$$\Rightarrow \frac{d^2y}{dx^2}=2\sec^2 2t\times\frac{1}{4at\cos 2t}=\frac{\sec^3 2t}{2at}.$$

OR Let $y=\tan^{-1}(a+bx)$ $\qquad\Rightarrow \dfrac{dy}{dx}=\dfrac{b}{1+(a+bx)^2}$

As the derivative of y takes the value 1 at x = 0 so, $\left.\dfrac{dy}{dx}\right]_{\text{at } x=0}=\dfrac{b}{1+(a+b\times 0)^2}=1$

$\Rightarrow 1=\dfrac{b}{1+a^2}$ $\qquad \therefore 1+a^2=b$.

Q17. We have $f(x)=\sin x-\cos x,\ 0<x<2\pi$ $\qquad\Rightarrow f'(x)=\cos x+\sin x,\ f''(x)=-\sin x+\cos x$

For local points of maxima and minima $f'(x)=\cos x+\sin x=0$

$\Rightarrow \tan x=-1$ $\qquad \therefore x=\dfrac{3\pi}{4},\dfrac{7\pi}{4}\in(0,\ 2\pi)$

$\because f''\left(\dfrac{3\pi}{4}\right)=\cos\dfrac{3\pi}{4}-\sin\dfrac{3\pi}{4}=-\sqrt{2}<0$ and, $f''\left(\dfrac{7\pi}{4}\right)=\cos\dfrac{7\pi}{4}-\sin\dfrac{7\pi}{4}=\sqrt{2}>0$

$\therefore f(x)$ is maximum at $x=\dfrac{3\pi}{4}$ and, minimum at $x=\dfrac{7\pi}{4}$

So, Local maximum value $f\left(\dfrac{3\pi}{4}\right)=\sin\dfrac{3\pi}{4}-\cos\dfrac{3\pi}{4}=\dfrac{1}{\sqrt{2}}-\left(-\dfrac{1}{\sqrt{2}}\right)=\sqrt{2}$

And, Local minimum value $f\left(\dfrac{7\pi}{4}\right)=\sin\dfrac{7\pi}{4}-\cos\dfrac{7\pi}{4}=-\dfrac{1}{\sqrt{2}}-\dfrac{1}{\sqrt{2}}=-\sqrt{2}$

OR Let a denote the side length of the triangle so, $\dfrac{da}{dt}=2\text{cms}^{-1}$.

Since area of the equilateral triangle is $A=\dfrac{\sqrt{3}\ a^2}{4}$ $\qquad\Rightarrow \dfrac{dA}{dt}=\dfrac{\sqrt{3}\ a}{2}\times\dfrac{da}{dt}$

$$\Rightarrow \left.\frac{dA}{dt}\right]_{\text{at } a=20\text{cm}}=\frac{\sqrt{3}\times 20}{2}\times 2=20\sqrt{3}\,\text{cm}^2\text{s}^{-1}.$$

Q18. Let $I=\displaystyle\int\frac{x^2-3x+1}{\sqrt{1-x^2}}dx$ $\qquad\Rightarrow I=-\displaystyle\int\frac{1-x^2+3x-2}{\sqrt{1-x^2}}dx$

$\Rightarrow I=-\left[\displaystyle\int\frac{1-x^2}{\sqrt{1-x^2}}dx+\int\frac{3x-2}{\sqrt{1-x^2}}dx\right]$ $\qquad\Rightarrow I=-\displaystyle\int\sqrt{1-x^2}dx-\int\frac{3x}{\sqrt{1-x^2}}dx+\int\frac{2}{\sqrt{1-x^2}}dx$

$$\Rightarrow I=-\frac{x}{2}\sqrt{1-x^2}-\frac{1}{2}\sin^{-1}x+\frac{3}{2}\int\frac{-2x}{\sqrt{1-x^2}}dx+2\sin^{-1}x$$

$\therefore I=-\frac{x}{2}\sqrt{1-x^2}+\frac{3}{2}\sin^{-1}x+\frac{3}{2}\left[2\sqrt{1-x^2}\right]+C$ i.e., $I=-\frac{x}{2}\sqrt{1-x^2}+\frac{3}{2}\sin^{-1}x+3\sqrt{1-x^2}+C$.

OR Let $I=\int(3-2x)\sqrt{2+x-x^2}\,dx$ Put $3-2x=A\frac{d}{dx}(2+x-x^2)+B=A(1-2x)+B$

On equating the coefficients of like terms, we get : A = 1, B = 2.

$\therefore I=\int[1(1-2x)+2]\sqrt{2+x-x^2}\,dx$ $\Rightarrow I=\int(1-2x)\sqrt{2+x-x^2}\,dx+2\int\sqrt{2+x-x^2}\,dx$

Put $2+x-x^2=t\Rightarrow(1-2x)dx=dt$ in 1^{st} integral

$$\therefore I=\int\sqrt{t}\,dt+2\int\sqrt{\left(\frac{3}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}\,dx$$

$$\Rightarrow I=\frac{2}{3}t^{3/2}+2\left[\frac{x-\frac{1}{2}}{2}\sqrt{\left(\frac{3}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}+\frac{9/4}{2}\sin^{-1}\left(\frac{x-\frac{1}{2}}{3/2}\right)\right]+C$$

$$\therefore I=\frac{2}{3}(2+x-x^2)^{3/2}+\left(x-\frac{1}{2}\right)\sqrt{2+x-x^2}+\frac{9}{4}\sin^{-1}\left(\frac{2x-1}{3}\right)+C.$$

Q19. $\int\frac{\sin 2x-\sin 2k}{\sin x-\sin k+\cos x-\cos k}dx=\int\frac{(1+\sin 2x)-(1+\sin 2k)}{\sin x-\sin k+\cos x-\cos k}dx$

$\Rightarrow\quad=\int\frac{(\cos x+\sin x)^2-(\cos k+\sin k)^2}{\sin x-\sin k+\cos x-\cos k}dx$

$\Rightarrow\quad=\int\frac{(\cos x+\sin x+\cos k+\sin k)(\cos x+\sin x-\cos k-\sin k)}{\sin x-\sin k+\cos x-\cos k}dx$

$\Rightarrow\quad=\int(\cos x+\sin x+\cos k+\sin k)dx=\sin x-\cos x+x(\cos k+\sin k)+C$

Q20. Let $I=\int\frac{\log x\,dx}{(x+1)^2}$ $\Rightarrow I=\log x\int\frac{dx}{(x+1)^2}-\int\left(\frac{d}{dx}\log x\int\frac{dx}{(x+1)^2}\right)dx$

$\Rightarrow I=-\frac{\log x}{(x+1)}+\int\frac{1}{x(x+1)}dx$ $\Rightarrow I=-\frac{\log x}{(x+1)}+\int\left(\frac{1}{x}-\frac{1}{x+1}\right)dx$

$\Rightarrow I=-\frac{\log x}{(x+1)}+\log|x|-\log|x+1|+C$ $\therefore I=\log\left|\frac{x}{x+1}\right|-\frac{\log x}{(x+1)}+C$.

Q21. Let $I=\int\frac{x^2+x+1}{(x^2+1)(x+2)}dx$ $\Rightarrow I=\int\left(\frac{x^2+1}{(x^2+1)(x+2)}+\frac{x}{(x^2+1)(x+2)}\right)dx$

$\Rightarrow I=\int\frac{1}{x+2}dx+\int\frac{x}{(x^2+1)(x+2)}dx$ $\Rightarrow I=\log|x+2|+\int\frac{x}{(x^2+1)(x+2)}dx\ \ldots$(i)

Consider $\frac{x}{(x^2+1)(x+2)}=\frac{A}{x+2}+\frac{2Bx}{x^2+1}+\frac{C}{x^2+1}$ $\Rightarrow x=A(x^2+1)+2Bx(x+2)+C(x+2)$

On equating the coefficients of like terms, we get : $A=-\frac{2}{5},B=\frac{1}{5},C=\frac{1}{5}$

$\therefore I=\log|x+2|-\frac{2}{5}\int\frac{dx}{x+2}+\frac{1}{5}\int\frac{2x\,dx}{x^2+1}+\frac{1}{5}\int\frac{dx}{x^2+1}$

$\Rightarrow I=\log|x+2|-\frac{2}{5}\log|x+2|+\frac{1}{5}\log|x^2+1|+\frac{1}{5}\tan^{-1}x+C$

Therefore, $I=\frac{3}{5}\log|x+2|+\frac{1}{5}\log|x^2+1|+\frac{1}{5}\tan^{-1}x+C$.

OR Let $I=\int\frac{x^3}{(x-1)(x^2+1)}dx$ $\Rightarrow I=\int\left(1+\frac{x^2-x+1}{(x-1)(x^2+1)}\right)dx$

$\Rightarrow I=\int 1\,dx+\int\left(\frac{x^2+1}{(x-1)(x^2+1)}-\frac{x}{(x-1)(x^2+1)}\right)dx$ $\Rightarrow I=x+\int\frac{1}{x-1}dx-\int\frac{x}{(x-1)(x^2+1)}dx$

$\Rightarrow I = x + \log|x-1| - I_1 \ldots(A)$ Now $I_1 = \int \frac{x}{(x-1)(x^2+1)}dx$

Consider $\frac{x}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{2Bx}{x^2+1} + \frac{C}{x^2+1}$ $\Rightarrow x = A(x^2+1) + 2Bx(x-1) + C(x-1)$

On equating the coefficients of like terms on both the sides, we get : $A = \frac{1}{2}, B = -\frac{1}{4}, C = \frac{1}{2}$.

$\therefore I = x + \log|x-1| - \left(\frac{1}{2}\int\frac{dx}{x-1} - \frac{1}{4}\int\frac{2xdx}{x^2+1} + \frac{1}{2}\int\frac{dx}{x^2+1}\right)$

$\Rightarrow I = x + \log|x-1| - \frac{1}{2}\log|x-1| + \frac{1}{4}\log|x^2+1| - \frac{1}{2}\tan^{-1}x + C$

Therefore, $I = x + \frac{1}{2}\log|x-1| + \frac{1}{4}\log|x^2+1| - \frac{1}{2}\tan^{-1}x + C$.

Q22. Consider $I = \int_{-\pi/2}^{\pi/2}\frac{\cos x}{1+e^x}dx \ldots(i)$ Using $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

$\Rightarrow I = \int_{-\pi/2}^{\pi/2}\frac{\cos\left(-\frac{\pi}{2}+\frac{\pi}{2}-x\right)}{1+e^{-\pi/2+\pi/2-x}}dx = \int_{-\pi/2}^{\pi/2}\frac{\cos x}{1+e^{-x}}dx$ $\Rightarrow I = \int_{-\pi/2}^{\pi/2}\frac{e^x\cos x}{e^x+1}dx \ldots(ii)$

On adding (i) & (ii), we get : $2I = \int_{-\pi/2}^{\pi/2}\frac{\cos x}{1+e^x}dx + \int_{-\pi/2}^{\pi/2}\frac{e^x\cos x}{e^x+1}dx$

$\Rightarrow 2I = \int_{-\pi/2}^{\pi/2}\left(\frac{1}{1+e^x} + \frac{e^x}{e^x+1}\right)\cos x dx$ $\Rightarrow 2I = \int_{-\pi/2}^{\pi/2}\cos x dx$

$\Rightarrow 2I = 2\int_0^{\pi/2}\cos x dx$ $\left[\because f(x) = \cos x = \cos(-x) = f(-x), \text{ i.e. } f \text{ is even function and, } \int_{-a}^{a}f(x)dx = 2\int_0^a f(x)dx, \text{ if } f \text{ is an even function}\right.$

$\therefore I = [\sin x]_0^{\pi/2} = \sin\frac{\pi}{2} - \sin 0 = 1$.

Q23. Let $I = \int_0^{\pi/4}\left(\frac{\sin x + \cos x}{3+\sin 2x}\right)dx$ Put $\sin x - \cos x = t \Rightarrow (\sin x + \cos x)dx = dt$

And, $(\sin x - \cos x)^2 = t^2 \Rightarrow 1 - t^2 = \sin 2x$. Also when $x = 0 \Rightarrow t = -1$ and when $x = \frac{\pi}{4} \Rightarrow t = 0$

$\therefore I = \int_{-1}^{0}\frac{dt}{4-t^2}$ $\Rightarrow I = \frac{1}{2(2)}\left[\log\left|\frac{2+t}{2-t}\right|\right]_{-1}^{0}$ $\Rightarrow I = \frac{1}{4}\left[\log\left|\frac{2+0}{2-0}\right| - \log\left|\frac{2-1}{2+1}\right|\right]$

$\Rightarrow I = \frac{1}{4}[0 + \log 3]$ $\therefore I = \frac{\log 3}{4}$.

OR Let $I = \int_0^{\pi/2}\frac{\cos^2 x dx}{1+3\sin^2 x}$ $\Rightarrow I = \int_0^{\pi/2}\frac{\sec^2 x dx}{\sec^4 x + 3\sec^2 x\tan^2 x}$ On dividing Nr & Dr by $\cos^4 x$

$\therefore I = \int_0^{\pi/2}\frac{\sec^2 x dx}{\sec^2 x(1+\tan^2 x + 3\tan^2 x)}$ $\Rightarrow I = \int_0^{\pi/2}\frac{\sec^2 x dx}{(1+\tan^2 x)(1+4\tan^2 x)}$

Put $\tan x = t \Rightarrow \sec^2 x dx = dt$. When $x = 0 \Rightarrow t = 0$ & when $x = \frac{\pi}{2} \Rightarrow t = \infty$

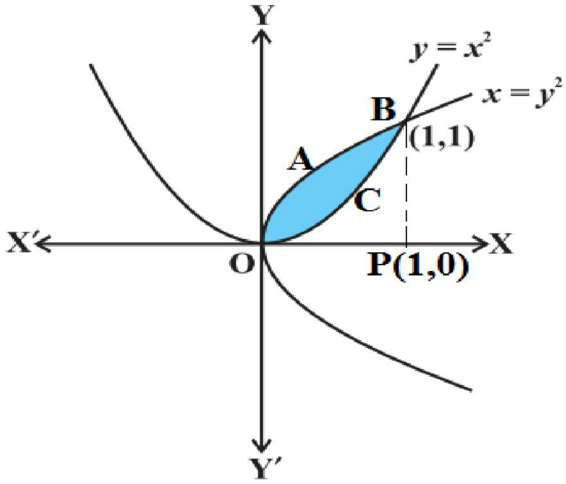
So, $I = \int_0^{\infty}\frac{dt}{(1+t^2)(1+4t^2)}$ $\Rightarrow I = -\frac{1}{3}\int_0^{\infty}\left(\frac{1}{1+t^2} - \frac{4}{1+4t^2}\right)dt$

$\Rightarrow I = -\frac{1}{3}\left[\tan^{-1}t - 4\frac{\tan^{-1}(2t)}{2}\right]_0^{\infty}$ $\therefore I = -\frac{1}{3}\left[\left(\frac{\pi}{2} - 2\times\frac{\pi}{2}\right) - (0-0)\right] = \frac{\pi}{6}$.

SECTION – C

Q24. Given curves are $y = x^2 \ldots$(i) and $y^2 = x \ldots$(ii)

On solving (i) and (ii), we get : $x = 0, 1$



Required ares, ar(OABCO) $= \int_0^1 [y_{ii} - y_i]dx$

$\Rightarrow \quad = \int_0^1 [\sqrt{x} - x^2]dx$

$\Rightarrow \quad = \left[\frac{2}{3}x^{3/2} - \frac{1}{3}x^3\right]_0^1$

$\Rightarrow \quad = \left[\frac{2}{3}\times 1^{3/2} - \frac{1^3}{3}\right] - [0-0] = \frac{1}{3}$ sq.units

Q25. Given $x^2 = 9p(9-y) \ldots$(i) and, $x^2 = p(y+1) \ldots$(ii)

By (i) & (ii), $p(y+1) = 9p(9-y) \quad \Rightarrow y = 8,\ x^2 = 9p \ldots$(A)

On diff. (i) & (ii) w.r.t. x both sides, $\frac{dy}{dx} = -\frac{2x}{9p}$ and $\frac{dy}{dx} = \frac{2x}{p}$

Since the curves (i) & (ii) cut each other at right angles so, $\left(\frac{2x}{p}\right)\left(-\frac{2x}{9p}\right) = -1 \quad \Rightarrow 4x^2 = 9p^2$

By (A), $4\times 9p = 9p^2 \qquad \Rightarrow p(p-4) = 0 \qquad \therefore p = 0, 4$

We shall reject $p = 0$ since it doesn't fit into the given conditions so, $p = 4$.

OR Given $xy = c^2 ...$(i)

Let $S = (ax + by) \qquad \Rightarrow S = ax + \frac{bc^2}{x} \qquad \Rightarrow \frac{dS}{dx} = a - \frac{bc^2}{x^2}$ and, $\frac{d^2S}{dx^2} = \frac{2bc^2}{x^3}$

For local points of maxima and/or minima, $\frac{dS}{dx} = a - \frac{bc^2}{x^2} = 0 \qquad \Rightarrow x = c\sqrt{\frac{b}{a}}$

$\because \left.\frac{d^2S}{dx^2}\right]_{\text{at } x = c\sqrt{\frac{b}{a}}} = \frac{2bc^2}{c^3\left(\frac{b}{a}\right)^{3/2}} > 0 \qquad \therefore S$ is minimum at $x = c\sqrt{\frac{b}{a}}$

Also, minimum value of $S = ax + by = 2ax \qquad \left[\because x = c\sqrt{\frac{b}{a}} \Rightarrow c^2 = \frac{ax^2}{b}\right.$ Replacing value of c^2 in (i), we get $\left. ax = by \right]$

That is, $S = 2ac\sqrt{\frac{b}{a}} \qquad \therefore S = 2c\sqrt{ab}$.

Dear students,

Keep all your best wishes and your biggest dreams close to your heart and dedicate time to them every day. If you truly care about what you do and you work diligently at it, there is almost nothing which you can't accomplish.

All the best for accomplishing your dreams. I'm certain that you all will do very well.

- O. P. Gupta, Math-Enthusiast