Topic: Chap 3 & 4 (Matrices & Determinants)

Basic Concepts

Matrices: A matrix is rectangular arrangement of elements(or functions) in order.

For example $A = \begin{bmatrix} 2 & 3 & x \\ 3 & 0 & 1 \end{bmatrix}$

Elements and Order of Matrix: In above matrix there are 2 rows and 3 columns. 1st row is named as R_1 consists of 3 elements $2 \quad 3 \quad x$ and 2nd row is named as R_2 consists of 3 elements $3 \quad 0 \quad 1$. 1st column is named as C_1 consists of 2 elements $\begin{matrix} 2 \\ 3 \end{matrix}$; 2nd column is named as C_2 consists of 2 elements $\begin{matrix} 3 \\ 0 \end{matrix}$ and 3rd column is named as C_3 consists of 2 elements $\begin{matrix} x \\ 1 \end{matrix}$.

A matrix with m rows and n columns is of order m X n such matrix is denoted by A_{mn}. For example in above matrix order of matrix A_{23} is 2X3.

General notation of element: a_{ij} denotes element at i^{th} $row\ and\ j^{th}\ place$. For ex. $a_{23} = 1$.

Example 1. For a 2 X 2 matrix A=[a_{ij}], whose elements are given by $a_{ij} = \frac{(i+2j)^2}{4}$, write the value of a_{21}. (DelhiC 2012)

Solution. Given $a_{ij} = \frac{(i+2j)^2}{4}$ to find a_{21} replace i and j by 2 and 1 respectively so $a_{21} = \frac{(2+2X1)^2}{4} = \frac{16}{4} = 4$.

Example 2. **Construct a matrix of order 2X2 whose elements are given by**

$$A_{ij} = \begin{cases} i^j & if\ i \geq j \\ i+j & if\ i < j \end{cases}$$

Solution. $A_{22} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ $as\ a_{11} = 1^1 = 1, a_{12} = 1+2 = 3, a_{21} = 2^1 = 2, a_{22} = 2^2 = 4;$

$$A_{22} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

Matrix Multiplication: Let A_{mn} and B_{pq} are 2 matrices then AXB can exists if number of columns in first matrix will be equal to number of rows in 2nd matrix like $n = p$ in this case.

Order of product AB will be mXq.

Let us do one example to explain matrix multiplication

$$(4 \quad 6)X\begin{pmatrix} 1 & 3 \\ 0 & -2 \end{pmatrix} = (4X1 + 6X0 \quad 4X3 + 6X(-2)) = (4 \quad 0)$$

In above example 1st matrix is of order 1X2 and 2nd matrix is of order 2X2 so product will exist as number of columns in 1st matrix is equal to number of rows in 2nd matrix and product will be of order 1X2. Each element of 1st row of Product has been evaluated by adding product of respective element of 1st row of 1st matrix and 1st column of 2nd matrix, and then 2nd column of 2nd matrix etc.

Operation on matrices: We have few examples to explain terms like comparison, addition, subtraction.

Example 3. If $2\begin{pmatrix}1 & 3\\ 0 & x\end{pmatrix}+\begin{pmatrix}y & 0\\ 1 & 2\end{pmatrix}=\begin{pmatrix}5 & 6\\ 1 & 8\end{pmatrix}$, *then write the value of* $(x+y)$.(DelhiC 2013)

Solution. Given $2\begin{pmatrix}1 & 3\\ 0 & x\end{pmatrix}+\begin{pmatrix}y & 0\\ 1 & 2\end{pmatrix}=\begin{pmatrix}5 & 6\\ 1 & 8\end{pmatrix}\Rightarrow\begin{pmatrix}2 & 6\\ 0 & 2x\end{pmatrix}+\begin{pmatrix}y & 0\\ 1 & 2\end{pmatrix}=\begin{pmatrix}5 & 6\\ 1 & 8\end{pmatrix}$

$$\Rightarrow\begin{pmatrix}2+y & 6\\ 1 & 2x+2\end{pmatrix}=\begin{pmatrix}5 & 6\\ 1 & 8\end{pmatrix}$$

After comparison of respective elements $2+y=5\Rightarrow y=3; 2x+2=8\Rightarrow x=3$

$\therefore x+y=3+3=6.$

Example 4. Solve the following matrix equation for x:

$$[x \quad 1]\begin{bmatrix}x & 0\\ -2 & 0\end{bmatrix}=0$$

Solution. Given $[x \quad 1]\begin{bmatrix}x & 0\\ -2 & 0\end{bmatrix}=0 \Rightarrow [x^2-2 \quad 0]=0\Rightarrow x^2-2=0\Rightarrow x=\pm\sqrt{2}$

Some important points:-

1. **Diagonal matrix** $a_{ij}=0, if\ i\neq j$ like $\begin{pmatrix}7 & 0 & 0\\ 0 & -4 & 0\\ 0 & 0 & 9\end{pmatrix}$

2. **Scalar matrix** $a_{ij}=\begin{cases}0, if\ i\neq j\\ k, if\ i=j\end{cases}$ like $\begin{pmatrix}7 & 0 & 0\\ 0 & 7 & 0\\ 0 & 0 & 7\end{pmatrix}$

3. **Identity matrix** $a_{ij}=\begin{cases}0, if\ i\neq j\\ 1, if\ i=j\end{cases}$ like $I_3=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$, $I_2=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

4. **Transpose of matrix:** If $A_{mn}=a_{ij}$ is a matrix then transpose of matrix is denoted by A^T or $A'=a_{ji}$. For ex. If $A=\begin{pmatrix}1 & 2 & 3\\ 4 & 0 & 5\end{pmatrix}$ then $A^T=\begin{pmatrix}1 & 4\\ 2 & 0\\ 3 & 5\end{pmatrix}$

5. Symmetric matrix : A is said to be Symmetric if A^T=A like $\begin{pmatrix}2 & 3\\ 3 & 4\end{pmatrix}$ is a symmetric matrix.
6. Skew Symmetric matrix: A is said to be Skew Symmetric if A^T= -A.

   For example $A=\begin{pmatrix}\mathbf{0} & \mathbf{3} & \mathbf{5}\\ \mathbf{-3} & \mathbf{0} & \mathbf{7}\\ \mathbf{-5} & \mathbf{-7} & \mathbf{0}\end{pmatrix}$ is a Skew Symmetric matrix.

   Explanation: $A^T=\begin{pmatrix}\mathbf{0} & \mathbf{-3} & \mathbf{-5}\\ \mathbf{3} & \mathbf{0} & \mathbf{-7}\\ \mathbf{5} & \mathbf{7} & \mathbf{0}\end{pmatrix}=-\begin{pmatrix}\mathbf{0} & \mathbf{3} & \mathbf{5}\\ \mathbf{-3} & \mathbf{0} & \mathbf{7}\\ \mathbf{-5} & \mathbf{-7} & \mathbf{0}\end{pmatrix}\Rightarrow A^T=-A$
7. Theorem: Every square matrix can be expressed as sum of Symmetric and Skew Symmetric matrices $A=\frac{1}{2}[(A+A^T)+(A-A^T)]$

   ↓ Symmetric ↓ Skew Sym
8. Properties of transpose of matrices:-
   (i) $(AB)^T=B^TA^T$
   (ii) $(A^T)^T=A$
   (iii) $(A+B)^T=A^T+B^T$
   (iv) $(kA)^T=kA^T$

Example 5. **For what value of k, the matrix** $\begin{pmatrix}\boldsymbol{k+3} & \mathbf{4} & \mathbf{5}\\ \mathbf{-4} & \mathbf{0} & \mathbf{-6}\\ \mathbf{-5} & \mathbf{6} & \boldsymbol{-k-3}\end{pmatrix}$ **is skew- symmetric?**

Solution. For skew matrix each element of main diagonal must be 0, so $\boldsymbol{k+3=0}\Rightarrow k=-3$.

Example 6. **If A =** $\begin{pmatrix}\boldsymbol{cos\alpha} & \boldsymbol{-sin\alpha}\\ \boldsymbol{sin\alpha} & \boldsymbol{cos\alpha}\end{pmatrix}$**, then for what value of α A+A′ is an identity matrix?** (NCERT)

Solution. Given **A+A′** = I $\Rightarrow\begin{pmatrix}\boldsymbol{cos\alpha} & \boldsymbol{-sin\alpha}\\ \boldsymbol{sin\alpha} & \boldsymbol{cos\alpha}\end{pmatrix}+\begin{pmatrix}\boldsymbol{cos\alpha} & \boldsymbol{sin\alpha}\\ \boldsymbol{-sin\alpha} & \boldsymbol{cos\alpha}\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

$$\Rightarrow\begin{pmatrix}2\boldsymbol{cos\alpha} & 0\\ 0 & 2\boldsymbol{cos\alpha}\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

i.e. $2\boldsymbol{cos\alpha}=\mathbf{1}\Rightarrow\boldsymbol{\alpha}=\frac{\pi}{3}$.

Example 7. To raise money for an orphanage, students of three schools A, B and C organized an exhibition in their locality, where they sold paper bags, scrap-books and pastel sheets made by them using recycled paper, at the rate of Rs. 10, Rs. 12 and Rs. 15 per unit respectively. School A sold 20 paper bags, 18 scrap-books and 30 pastel sheets. School B sold 25 paper bags 15, scrap-books and 25 pastel sheets while School C sold 6 paper bags, 15 scrap-books and 35 pastel sheets. Using matrices, find the total amount raised by each school.

By such exhibition, which values are inculcated in the students?

Solution. Sale matrix of school A, B and C is given by $\begin{pmatrix} 20 & 18 & 30 \\ 25 & 15 & 25 \\ 6 & 15 & 35 \end{pmatrix}$ and price matrix by $\begin{pmatrix} 10 \\ 12 \\ 15 \end{pmatrix}$

Total amount raised by each school is given by

$$\begin{pmatrix} 20 & 18 & 30 \\ 25 & 15 & 25 \\ 6 & 15 & 35 \end{pmatrix} X \begin{pmatrix} 10 \\ 12 \\ 15 \end{pmatrix} = \begin{pmatrix} 866 \\ 805 \\ 765 \end{pmatrix}$$

School A= Rs. 866, School B= Rs. 805, School C= Rs. 765

Values: Helping the orphans, use of recycled paper.

Example 8. Express the following matrix as the sum of a symmetric matrix and skew symmetric matrix and verify your result:

$\begin{pmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{pmatrix}$(AI 2010)

Solution. Let A= $\begin{pmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{pmatrix} \Rightarrow A' = \begin{pmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{pmatrix}$

$$\text{Let } P = \frac{1}{2}(A + A') = \frac{1}{2}\left\{\begin{pmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{pmatrix} + \begin{pmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{pmatrix}\right\} = \begin{pmatrix} 3 & 1/2 & -5/2 \\ 1/2 & -2 & -2 \\ -5/2 & -2 & 2 \end{pmatrix}$$

$$\Rightarrow P' = \begin{pmatrix} 3 & 1/2 & -5/2 \\ 1/2 & -2 & -2 \\ -5/2 & -2 & 2 \end{pmatrix} \Rightarrow P' = P$$

$\therefore P$ is a symmetric matrix.

$$\text{And let } Q = \frac{1}{2}(A - A') = \frac{1}{2}\left\{\begin{pmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{pmatrix} - \begin{pmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{pmatrix}\right\} = \begin{pmatrix} 0 & -5/2 & -3/2 \\ 5/2 & 0 & -3 \\ 3/2 & 3 & 0 \end{pmatrix}$$

$$\Rightarrow Q' = \begin{pmatrix} 0 & -5/2 & -3/2 \\ 5/2 & 0 & -3 \\ 3/2 & 3 & 0 \end{pmatrix} \Rightarrow Q' = -\begin{pmatrix} 0 & 5/2 & 3/2 \\ -5/2 & 0 & 3 \\ -3/2 & -3 & 0 \end{pmatrix} = -Q$$

$\therefore Q$ is a skew symmetric matrix.

$$\text{Now } P + Q = \begin{pmatrix} 3 & 1/2 & -5/2 \\ 1/2 & -2 & -2 \\ -5/2 & -2 & 2 \end{pmatrix} + \begin{pmatrix} 0 & -5/2 & -3/2 \\ 5/2 & 0 & -3 \\ 3/2 & 3 & 0 \end{pmatrix} = \begin{pmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{pmatrix}$$ is given matrix.

Elementary transformations of a Matrix: There are 3 basic transformations due to rows and 3 basic transformations due to columns.

Elementary row transformations:-

(i) Interchange of any two rows:

Like if you want to interchange first two rows of $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ to have identity matrix, you will perform $R_1 \leftrightarrow R_2$ to have $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(ii) Multiplication of the elements of any row by any non zero number:

Like if you want to transfer $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ to have identity matrix, you will perform $R_1 \rightarrow \frac{1}{2}R_1$ to have $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(iii) Corresponding addition(or subtraction) of elements of 2 rows:

Like if you want to transfer $\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ to have identity matrix, you will perform $R_2 \rightarrow R_2 - 3R_1$ to have $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

How to find inverse of given matrix using elementary row transformations: Let A be the given matrix then write A=IA and apply elementary row operations on left matrix A and I to convert left matrix A as identity matrix and right matrix A will remain unchanged.

Then we have I=BA

So $A^{-1} = B$.

Example 9. Use elementary row operation $R_2 \rightarrow R_2 - 3R_1$ in the matrix equation $\begin{pmatrix} 12 & -3 \\ 26 & -5 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 5 & -2 \end{pmatrix}$.

Solution. Operation will be performed in left matrix and first matrix in right side 2nd matrix in right side will remain unchanged.

Now elementary row operation $R_2 \rightarrow R_2 - 3R_1$ will result as

$$\begin{pmatrix} 12 & -3 \\ -10 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 5 & -2 \end{pmatrix}$$

Example 10. Using elementary row operations, find the inverse of the matrix $\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$(Delhi 2010)

Solution. Let A= $\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$

Now A= IA

$$\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} A$$

Now perform row operation to make $a_{11} = 1$, you can perform $R_1 \to \frac{1}{2} R_1$ or $R_1 \to R_1 - R_2$

We will perform $R_1 \to R_1 - 2R_2$ to avoid fraction

$$\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} A$$

Now perform row operation to make $a_{21} = 0$, with the help of a_{11} you can perform $R_2 \to R_2 - R_1$

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} A$$

Now perform row operation to make $a_{22} = 1$ in this case we have already $a_{22} = 1$ so we will do next step.

Now perform row operation to make $a_{12} = 0$, with the help of a_{21} you can perform $R_1 \to R_1 - 2R_2$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix} A$$

$$\therefore A^{-1} = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$$

Example 11. Using elementary row operations, find the inverse of the matrix $\begin{pmatrix} 3 & 5 \\ 8 & 13 \end{pmatrix}$

Solution. Let A= $\begin{pmatrix} 3 & 5 \\ 8 & 13 \end{pmatrix}$

Now A= IA

$$\begin{pmatrix} 3 & 5 \\ 8 & 13 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} A$$

Now perform row operation to make $a_{11} = 1$, you can perform $R_1 \to \frac{1}{3} R_1$

$$\begin{pmatrix} 1 & \frac{5}{3} \\ 8 & 13 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{pmatrix} A$$

Now perform row operation to make $a_{21} = 0$, with the help of a_{11} you can perform $R_2 \to R_2 - 8R_1$

$$\begin{pmatrix} 1 & \frac{5}{3} \\ 0 & -\frac{1}{3} \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & 0 \\ -\frac{8}{3} & 1 \end{pmatrix} A$$

Now perform row operation to make $a_{22} = 1$, you can perform $R_2 \to -3\ R_2$

$$\begin{pmatrix} 1 & \frac{5}{3} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & 0 \\ 8 & -3 \end{pmatrix} A$$

Now perform row operation to make $a_{12} = 0$, with the help of a_{21} you can perform $R_1 \to R_1 - \frac{5}{3} R_2$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -13 & 5 \\ 8 & -3 \end{pmatrix} A$$

$$\therefore A^{-1} = \begin{pmatrix} -13 & 5 \\ 8 & -3 \end{pmatrix}$$

Elementary column transformations:-

(i) Interchange of any two columns:

Like if you want to interchange first two columns of $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ to have identity matrix, you will perform $C_2 \leftrightarrow C_3$ to have $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(ii) Multiplication of the elements of any column by any non zero number:

Like if you want to transfer $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ to have identity matrix, you will perform $C_1 \to \frac{1}{2} C_1$ to have $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(iii) Corresponding addition of elements of 2 columns:

Like if you want to transfer $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{pmatrix}$ to have identity matrix, you will perform $C_2 \to C_2 - 4R_3$ to have $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

How to find inverse of given matrix using elementary column transformations:

Let A be the given matrix then write A = AI and apply elementary column operations on left matrix A and I to convert left matrix A as identity matrix and right matrix A will remain unchanged.

Then we have I=AB

So $A^{-1} = B$.

Example 12. Use elementary column operation $C_2 \to C_2 - 2C_1$ in the matrix equation $\begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$.

Solution. Operation will be performed in left matrix and 2nd matrix in right side 1st matrix in right side will remain unchanged.

Now elementary row operation $C_2 \to C_2 - 2C_1$ will result as

$$\begin{pmatrix} 4 & -6 \\ 3 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 2 & -4 \\ 1 & -1 \end{pmatrix}$$

Example 13. Using elementary column operations, find the inverse of the matrix $\begin{pmatrix} 3 & 5 \\ 4 & 7 \end{pmatrix}$

Solution. Let A= $\begin{pmatrix} 3 & 5 \\ 4 & 7 \end{pmatrix}$

Now A= AI

$$\begin{pmatrix} 3 & 5 \\ 4 & 7 \end{pmatrix} = A\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Now perform column operation to make $a_{11} = 1$, you can perform $C_1 \to \frac{1}{3}C_1$

$$\begin{pmatrix} 1 & 5 \\ \frac{4}{3} & 7 \end{pmatrix} = A\begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{pmatrix}$$

Now perform column operation to make $a_{12} = 0$, with the help of a_{11} you can perform $C_2 \to C_2 - 5C_1$

$$\begin{pmatrix} 1 & 0 \\ \frac{4}{3} & \frac{1}{3} \end{pmatrix} = A\begin{pmatrix} \frac{1}{3} & -\frac{5}{3} \\ 0 & 1 \end{pmatrix}$$

Now perform column operation to make $a_{22} = 1$, you can perform $C_2 \to 3C_2$

$$\begin{pmatrix} 1 & 0 \\ \frac{4}{3} & 1 \end{pmatrix} = A\begin{pmatrix} \frac{1}{3} & -5 \\ 0 & 3 \end{pmatrix}$$

Now perform column operation to make $a_{21} = 0$, with the help of a_{22} you can perform $C_1 \to C_1 - \frac{4}{3}C_2$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = A\begin{pmatrix} 7 & -5 \\ -4 & 3 \end{pmatrix}$$

$$\therefore A^{-1} = \begin{pmatrix} 7 & -5 \\ -4 & 3 \end{pmatrix}$$

Example 14. Find the inverse of the following matrix using elementary operations:

A=$\begin{pmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix}$(AI 2010)

Solution. In this problem you can use either of two elementary operations as instruction has not been given. We will apply elementary row operation.

Let A=$\begin{pmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix}$

Now A= IA

$$\begin{pmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A$$

Here always try to get 2 zeroes in any one column and then use the rows with these two zeroes to reduce the given matrix as identity matrix.

To make 2 zeroes in C_1. Now perform row operation to make $a_{21} = 0$, with the help of a_{11} you can perform $R_2 \to R_2 + R_1$

$$\begin{pmatrix} 1 & 2 & -2 \\ 0 & 5 & -2 \\ 0 & -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A$$

Now perform row operation to make $a_{23} = 0$, with the help of a_{33} you can perform $R_2 \to R_2 + 2R_3$

$$\begin{pmatrix} 1 & 2 & -2 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} A$$

Now perform row operation to make $a_{32} = 0$, with the help of a_{22} you can perform $R_3 \to R_3 + 2R_2$

$$\begin{pmatrix} 1 & 2 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{pmatrix} A$$

Now perform row operation to make $a_{12} = 0$, with the help of a_{21} you can perform $R_1 \to R_1 - 2R_2$

$$\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -2 & -4 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{pmatrix} A$$

Now perform row operation to make $a_{13} = 0$, with the help of a_{31} you can perform $R_1 \to R_1 + 2R_3$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{pmatrix} A$$

$$\therefore A^{-1} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

Note: (a) In elementary transformations you have to use either row transformations or column transformation in finding inverse of any given matrix.

(b) You can not use both transformations to find inverse of given matrix.

Example 15. For the following matrices A and B, verify that (AB)'=B'A'

A = $\begin{pmatrix} 2 \\ 5 \\ 6 \end{pmatrix}$, B =$(3 \quad 2 \quad -1)$

Solution. AB = $\begin{pmatrix} 2 \\ 5 \\ 6 \end{pmatrix} X\,(3 \quad 2 \quad -1) = \begin{pmatrix} 6 & 4 & -2 \\ 15 & 10 & -5 \\ 18 & 12 & -6 \end{pmatrix} \Rightarrow \text{LHS} = (\text{AB})' = \begin{pmatrix} 6 & 15 & 18 \\ 4 & 10 & 12 \\ -2 & -5 & -6 \end{pmatrix}$

For RHS B'= $\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$ and A'=$(2 \quad 5 \quad 6) \Rightarrow \text{RHS} = \text{B'A'} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} X (2 \quad 5 \quad 6) = \begin{pmatrix} 6 & 15 & 18 \\ 4 & 10 & 12 \\ -2 & -5 & -6 \end{pmatrix}$

$\therefore$ LHS = RHS.

Example 16. Let A = $\begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}$, B = $\begin{pmatrix} 5 & 2 \\ 7 & 4 \end{pmatrix}$, C = $\begin{pmatrix} 2 & 5 \\ 3 & 8 \end{pmatrix}$. Find a matrix D such that CD-AB=0. (NCERT)

Solution. As per compatibility of matrix multiplication and addition D will be square matrix of order 2.

Let us assume D = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ Now given CD-AB=0 $\Rightarrow \begin{pmatrix} 2 & 5 \\ 3 & 8 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 5 & 2 \\ 7 & 4 \end{pmatrix} = 0$

$$\Rightarrow \begin{pmatrix} 2a+5c & 2b+5d \\ 3a+8c & 3b+8d \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 43 & 22 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 2a+5c-3 & 2b+5d \\ 3a+8c-43 & 3b+8d-22 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

By matrix comparison $2a+5c=3$ and $3a+8c=43 \Rightarrow a=-191, c=77$

$$2b+5d=0 \text{ and } 3b+8d=22 \Rightarrow b=-110, d=44$$

$\therefore$ D = $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -191 & -110 \\ 77 & 44 \end{pmatrix}$

9. **Determinant:** It is a number associated to a square matrix. Like if A= $\begin{pmatrix} 4 & 5 \\ 3 & 2 \end{pmatrix}$ is a matrix then determinant of A is denoted by |A| and |A|= 4X2-5X3= -7

Now take a 3X3 matrix B=$\begin{pmatrix}1 & 1 & -2\\ 2 & 1 & -3\\ 5 & 4 & -9\end{pmatrix}$ then |B| can be evaluated in either of 6 ways.

Sign convention for determinant evaluation is $\begin{vmatrix}+ & - & +\\ - & + & -\\ + & - & +\end{vmatrix}$

If you wish to evaluate along R_1 then

$$|B| = a_{11}\begin{vmatrix}a_{22} & a_{23}\\ a_{32} & a_{33}\end{vmatrix} - a_{12}\begin{vmatrix}a_{21} & a_{23}\\ a_{31} & a_{33}\end{vmatrix} + a_{13}\begin{vmatrix}a_{21} & a_{22}\\ a_{31} & a_{32}\end{vmatrix}$$

$$= 1X\begin{vmatrix}1 & -3\\ 4 & -9\end{vmatrix} - 1X\begin{vmatrix}2 & -3\\ 5 & -9\end{vmatrix} + (-2)X\begin{vmatrix}2 & 1\\ 5 & 4\end{vmatrix}$$

= 3-(-3)+(-2)X3

= 0

10. Minor M_{12} in $\begin{bmatrix}1 & 3 & 6\\ 7 & 6 & -8\\ 9 & -7 & 5\end{bmatrix}$ = $\begin{vmatrix}7 & -8\\ 9 & 5\end{vmatrix}$ = $7X5 - 9X(-8) = 107$

11. Cofactor A_{12} in $\begin{bmatrix}1 & 3 & 6\\ 7 & 6 & -8\\ 9 & -7 & 5\end{bmatrix}$ =$(-1)^{1+2}\begin{vmatrix}7 & -8\\ 9 & 5\end{vmatrix}$ = $-[7X5 - 9X(-8)] = -107$

12. Adjoint of matrix adj A=$(C_{ij})^T$ where C_{ij} denotes the cofactor of a_{ij}

13. Properties of adjoint of square matrices:-
    - (i) adj(AB)=adjB X adjA
    - (ii) $adjA^T=(adjA)^T$
    - (iii) A(adjA)=|A|I=(adjA)A
    - (iv) $|adjA|=|A|^{n-1}$ where n is order of square matrix.

14. Singular or non invertible matrix: A is said to be Singular or non invertible matrix if |A|=0

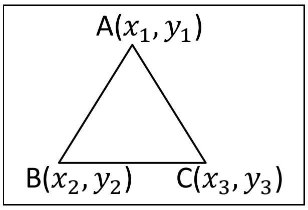
15. Non Singular or invertible matrix: A is said to be Non Singular or invertible matrix if $|A| \neq 0$

16. Inverse of matrix: If AB=I then A and B are inverse of each other like $A^{-1}=B; B^{-1}=A$

17. Inverse of matrix A is given by $A^{-1} = \frac{adjA}{|A|}$

18. Properties of Inverse of Matrices:-
    - (i) $(AB)^{-1}=B^{-1}A^{-1}$
    - (ii) $(A^T)^{-1}=(A^{-1})^T$
    - (iii) $(A^{-1})^{-1}=A$
    - (iv) $|A^{-1}| = \frac{1}{|A|}$

19. Area of a triangle : In co-ordinate geometry we have learnt area of a triangle with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3) is given by $\frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$. Area of this triangle can be given using determinant as



$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$ which gives the same expression as above used for the area.

➧ Note: (i) To check co linearity of three points equate area of triangle to zero.

(ii) To find equation of straight line using determinant passing through given two points take the third point as (x,y) then apply condition of co linearity of three points.

(iii) Always take the absolute value of determinant as area is positive.

(iv) If area is given then use positive and negative both values of determinant for calculation.

Example 17. Find the value of K so that the matrix $\begin{bmatrix} 2+3k & 2 \\ 2 & 5 \end{bmatrix}$ is singular.

Solution. For singular or non invertible matrix $\begin{vmatrix} 2+3k & 2 \\ 2 & 5 \end{vmatrix} = 0 \Rightarrow 5(2+3\mathrm{k}) - 4 = 0 \Rightarrow \mathrm{k} = -\frac{2}{5}$

Example 18. If $\begin{vmatrix} sin\alpha & cos\beta \\ cos\alpha & sin\beta \end{vmatrix} = \frac{\sqrt{3}}{2}$, **where α, β are acute angles, then write the value of α+β.**

Solution. $\begin{vmatrix} sin\alpha & cos\beta \\ cos\alpha & sin\beta \end{vmatrix} = \frac{\sqrt{3}}{2} \Rightarrow sin\alpha\, sin\beta - cos\alpha cos\beta = \frac{\sqrt{3}}{2} \Rightarrow -(cos\alpha cos\beta - sin\alpha\, sin\beta) = \frac{\sqrt{3}}{2}$

$$\Rightarrow \cos(\alpha + \beta) = -\frac{\sqrt{3}}{2} \Rightarrow \alpha + \beta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

Example 19. If A is a square matrix of order 3 such that | A|= 5, find |adj A|. (AIC 2013)

Solution. We know that |adj A|=|A|$^{n-1}$ where n is order of square matrix

$\Rightarrow$ |adj A|= 5^{3-1} = 5^2 = 25.

Example 20. If A is a square matrix of order 3 such that |adj A|= 256, find |A|.

Solution. We know that |adj A|=|A|$^{n-1}$ where n is order of square matrix $\Rightarrow 256 = |\mathrm{A}|^{3-1}$

$$\Rightarrow 256 = |\mathrm{A}|^2 \Rightarrow |\mathrm{A}| = \pm 16$$

Example 21. If A and B are matrices of order 3 and |A|= 5, |B|= 3, then find |3AB|.(NCERT Exemplar)

Solution. |3AB| = 3^3|AB|= 27|A||B|= 27 X 5 X 3 = 405.

Example 22. Let A = $\begin{pmatrix} 2 & 3 \\ -1 & 2 \end{pmatrix}$. Then show that $A^2 - 4A + 7I = 0$. Using this result calculate A^5 also. (NCERT Exemplar)

Solution. We have A = $\begin{pmatrix} 2 & 3 \\ -1 & 2 \end{pmatrix} \Rightarrow A^2 = \begin{pmatrix} 2 & 3 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 12 \\ -4 & 1 \end{pmatrix}, 4A = \begin{pmatrix} 8 & 12 \\ -4 & 8 \end{pmatrix}$

$$\therefore A^2 - 4A + 7I = \begin{pmatrix} 1 & 12 \\ -4 & 1 \end{pmatrix} - \begin{pmatrix} 8 & 12 \\ -4 & 8 \end{pmatrix} + \begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix} = \begin{pmatrix} 1-8+7 & 12-12+0 \\ -4+4+0 & 1-8+7 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$\therefore A^2 - 4A + 7I = 0 \Rightarrow A^2 = 4A - 7I \Rightarrow AA^2 = A(4A - 7I)$ (Pre multiply by A)

$\Rightarrow A^3 = 4A^2 - 7A \Rightarrow A^2A^3 = A^2\{4(4A-7I) - 7A\} \Rightarrow A^5 = 9A^3 - 28A^2 = 9(4A^2 - 7A) - 28A^2$

$\Rightarrow A^5 = 8A^2 - 63A = 8A^2 - 63A = 8(4A - 7I) - 63A = -31A - 56I$

$$\Rightarrow A^5 = -31\begin{pmatrix} 2 & 3 \\ -1 & 2 \end{pmatrix} - 56\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -118 & -93 \\ 31 & -118 \end{pmatrix}$$

Example 23. If A = $\begin{bmatrix} 1 & 4 & 0 \\ 0 & 2 & 4 \\ 6 & -3 & 0 \end{bmatrix}$ then evaluate |A.(adjA)|.

Solution. $\begin{vmatrix} -7 & 4 & 0 \\ 0 & 2 & 1 \\ 6 & -3 & 0 \end{vmatrix} = -1\begin{vmatrix} -7 & 4 \\ 6 & -3 \end{vmatrix} = -(21-24) = 3$

We know that $|\text{adj } A| = |A|^{n-1}$ where n is order of square matrix

and $|A.(\text{adj}A)| = |A|\,|\text{adj } A| = |A|\,|A|^{n-1} = |A|^n = |A|^3 = 3^3 = 27$

Example 24. The value of the determinant of a matrix A of order 3X3 is 4. Find the value of |3A|. (AIC 2012)

Solution. We know that $|3A| = 3^3|A| = 27 \times 4 = 108$.

Example 25. If the value of a third order determinant is 12, then find the value of the determinant formed by replacing each element by its co-factor. (NCERT Exemplar)

Solution. $\because$ adj $A = (C_{ij})^T$ where C_{ij} denotes the cofactor of a_{ij} and $(A^T)^T = A$

$\Rightarrow$ Required determinant = $|(\text{adj } A)^T| = |\text{adj } A| = |A|^{n-1} = 12^{3-1} = 12^2 = 144$.

Example 26. Find the equation of the line joining A(2,-6) and B(5,4) using determinants and find the value of k if D(k,4) is a point such that area of triangle ABD is 35 square units.

Solution. Let $C(x, y)$ be a point on AB. Then A, B and C are co linear and hence area of triangle ABC is 0.

Area of triangle using determinant is given by $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 2 & -6 & 1 \\ 5 & 4 & 1 \\ x & y & 1 \end{vmatrix} = 0$

$\Rightarrow \frac{1}{2}[x(-6-4) - y(2-5) + 1(8+30)] = \frac{1}{2}[-10x + 3y + 38] = 0$

∴ Required equation of straight line is $-10x + 3y + 38.$

Given area of Δ ABD = 35 square units

$\Rightarrow \frac{1}{2}\begin{vmatrix} 2 & -6 & 1 \\ 5 & 4 & 1 \\ k & 4 & 1 \end{vmatrix} = \pm 35 \Rightarrow \mathrm{k}(-10) - 4(-3) + 1(38) = \pm 70 \Rightarrow -10\mathrm{k} = \pm 70 - 12 - 38$

$\Rightarrow \mathrm{k} = -2, 12.$

Example 27. Using properties of determinants, prove the following :

$\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix}$=(a-b)(b-c)(c-a)(ab + bc + ca) (AIC 2013)

Solution. $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix} = \frac{1}{abc}\begin{vmatrix} a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \\ abc & bca & abc \end{vmatrix}$ (On applying $C_1 \to aC_1, C_2 \to bC_2$ and $C_3 \to cC_3$)

On taking abc common from R_1

$= \frac{abc}{abc}\begin{vmatrix} a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \\ abc & bca & abc \end{vmatrix} = \begin{vmatrix} a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \\ 1 & 1 & 1 \end{vmatrix}$

$= \begin{vmatrix} a^2-b^2 & b^2-c^2 & c^2 \\ a^3-b^3 & b^3-c^3 & c^3 \\ 0 & 0 & 1 \end{vmatrix}$ (On applying $C_1 \to C_1 - C_2$ & $C_2 - C_3$) $= \begin{vmatrix} a^2-b^2 & b^2-c^2 \\ a^3-b^3 & b^3-c^3 \end{vmatrix}$

$= (a-b)(b-c)\begin{vmatrix} a+b & b+c \\ a^2+ab+b^2 & b^2+bc+c^2 \end{vmatrix}$ (On taking a-b and b-c common from C_1 and C_2)

$= (a-b)(b-c)\begin{vmatrix} a+b & c-a \\ a^2+ab+b^2 & bc+c^2-a^2-ab \end{vmatrix}$ (On applying $C_1 \to C_1 - C_2$)

$= (a-b)(b-c)(c-a)\begin{vmatrix} a+b & 1 \\ a^2+ab+b^2 & a+b+c \end{vmatrix}$ (On taking c-a common from C_2)

$= (\mathrm{a}-\mathrm{b})(\mathrm{b}-\mathrm{c})(\mathrm{c}-\mathrm{a})(\mathrm{ab} + \mathrm{bc} + \mathrm{ca})$

Example 28. Prove the following, using properties of determinants:

$\begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix}$ = $a^3 + b^3 + c^3 - 3abc$ (DelhiC 2012)

Solution. $\begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix} = \begin{vmatrix} a+b+c & b & c \\ 0 & b-c & c-a \\ 2(a+b+c) & c+a & a+b \end{vmatrix}$ (On applying $C_1 \to C_1 + C_2 + C_3$)

$= (a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & b-c & c-a \\ 2 & c+a & a+b \end{vmatrix}$ (On taking $a+b+c$ common from C_1)

$= (a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & b-c & c-a \\ 0 & c+a-2b & a+b-2c \end{vmatrix}$ (On applying $R_3 \to R_3 - 2R_1$)

$= (a+b+c)\begin{vmatrix} b-c & c-a \\ c+a-2b & a+b-2c \end{vmatrix} = (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$

$= a^3 + b^3 + c^3 - 3abc$

Example 29. Prove the following, using properties of determinants:

$$\begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix} = 2\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}$$

Solution. $\begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix} = \begin{vmatrix} 2(a+b+c) & 2(p+q+r) & 2(x+y+z) \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix}$ $(R_1 \to R_1 + R_2 + R_3)$ $= 2\begin{vmatrix} (a+b+c) & (p+q+r) & (x+y+z) \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix} = 2\begin{vmatrix} b & q & y \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix}$ $(R_1 \to R_1 - R_2)$

$= 2\begin{vmatrix} b & q & y \\ c+a & r+p & z+x \\ a & p & x \end{vmatrix}$ $(R_3 \to R_3 - R_1)$ $= 2\begin{vmatrix} b & q & y \\ c & r & z \\ a & p & x \end{vmatrix}$ $(R_2 \to R_2 - R_3)$

$= -2\begin{vmatrix} b & q & y \\ a & p & x \\ c & r & z \end{vmatrix}$ $(R_2 \leftrightarrow R_3)$ $= 2\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}$

Example 30. Prove the following, using properties of determinants :

$$\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = (1+a^2+b^2)^3$$ (Foreign 2013)

Solution. $\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix}$

(On applying $C_1 \to C_2 + a\,C_3$)

$$=\begin{vmatrix} 1+a^2-b^2 & 0 & -2b \\ 2ab & 1+a^2+b^2 & 2a \\ 2b & -a(1+a^2+b^2) & 1-a^2-b^2 \end{vmatrix} = (1+a^2+b^2)\begin{vmatrix} 1+a^2-b^2 & 0 & -2b \\ 2ab & 1 & 2a \\ 2b & -a & 1-a^2-b^2 \end{vmatrix}$$

(On applying $R_3 \to R_3 + a\,R_2$)

$$= (1+a^2+b^2)\begin{vmatrix} 1+a^2-b^2 & 0 & -2b \\ 2ab & 1 & 2a \\ 2b+2a^2b & 0 & 1+a^2-b^2 \end{vmatrix} = (1+a^2+b^2)\begin{vmatrix} 1+a^2-b^2 & -2b \\ 2b+2a^2b & 1+a^2-b^2 \end{vmatrix}$$

$$= (1+a^2+b^2)\left[(1+a^2-b^2)^2 + 2b(2b+2a^2b)\right]$$

$$= (1+a^2+b^2)(1+(a^2)^2+(b^2)^2+2a^2+2b^2+2a^2b^2) = (1+a^2+b^2)(1+a^2+b^2)^2$$

$$= (1+a^2+b^2)^3$$

Example 31. Prove the following, using properties of determinants:

$$\begin{vmatrix} a+bx^2 & c+dx^2 & p+qx^2 \\ ax^2+b & cx^2+d & px^2+q \\ u & v & w \end{vmatrix} = (x^4-1)\begin{vmatrix} b & d & q \\ a & c & p \\ u & v & w \end{vmatrix} \text{(Foreign 10)}$$

Solution. $\begin{vmatrix} a+bx^2 & c+dx^2 & p+qx^2 \\ ax^2+b & cx^2+d & px^2+q \\ u & v & w \end{vmatrix}$ (On applying $R_1 \to R_1 + R_2$)

$$= \begin{vmatrix} (a+b)(1+x^2) & (c+d)(1+x^2) & (p+q)(1+x^2) \\ ax^2+b & cx^2+d & px^2+q \\ u & v & w \end{vmatrix} = (1+x^2)\begin{vmatrix} (a+b) & (c+d) & (p+q) \\ ax^2+b & cx^2+d & px^2+q \\ u & v & w \end{vmatrix}$$

$$= (1+x^2)\begin{vmatrix} a & c & p \\ ax^2+b & cx^2+d & px^2+q \\ u & v & w \end{vmatrix} + (1+x^2)\begin{vmatrix} b & d & q \\ ax^2+b & cx^2+d & px^2+q \\ u & v & w \end{vmatrix}$$ (On splitting R_1)

$$= (1+x^2)\begin{vmatrix} a & c & p \\ b & d & q \\ u & v & w \end{vmatrix} + (1+x^2)\begin{vmatrix} b & d & q \\ ax^2 & cx^2 & px^2 \\ u & v & w \end{vmatrix} (R_2 \to R_2 - x^2R_1, R_2 \to R_2 - R_1)$$

$$= -(1+x^2)\begin{vmatrix} b & d & q \\ a & c & p \\ u & v & w \end{vmatrix} + (1+x^2)x^2\begin{vmatrix} b & d & q \\ a & c & p \\ u & v & w \end{vmatrix} (R_1 \leftrightarrow R_2, Taking\ x^2 common\ from\ R_2)$$

$$= (1+x^2)\begin{vmatrix} b & d & q \\ a & c & p \\ u & v & w \end{vmatrix}(x^2-1) = (x^4-1)\begin{vmatrix} b & d & q \\ a & c & p \\ u & v & w \end{vmatrix}$$

Example 32. Find the maximum value of the determinant $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+\sin\theta & 1 \\ 1 & 1 & 1+\cos\theta \end{vmatrix}$.

(NCERT Exemplar)

Solution. $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+\sin\theta & 1 \\ 1 & 1 & 1+\cos\theta \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & \sin\theta & 0 \\ 1 & 0 & \cos\theta \end{vmatrix}$ $(On\ C_2 \to C_2 - C_1\ and\ C_3 \to C_3 - C_1)$

$= \sin\theta\cos\theta = \frac{1}{2}\sin 2\theta = \frac{1}{2}$ $(As\ maximum\ value\ of\ \sin 2\theta = 1)$

Example 33. There are two values of a which makes determinant, $\Delta = \begin{vmatrix} 1 & -2 & 5 \\ 2 & a & -1 \\ 0 & 4 & 2a \end{vmatrix} = 86$, then find the sum these values. (NCERT Exemplar)

Solution. $\begin{vmatrix} 1 & -2 & 5 \\ 2 & a & -1 \\ 0 & 4 & 2a \end{vmatrix} = 86 \Rightarrow \begin{vmatrix} a & -1 \\ 4 & 2a \end{vmatrix} - 2\begin{vmatrix} -2 & 5 \\ 4 & 2a \end{vmatrix} = 86 \Rightarrow 2a^2 + 4 - 2(-4a - 20) = 86$

$\Rightarrow 2a^2 + 8a + 24 - 86 = 0 \Rightarrow a^2 + 4a - 31 = 0 \Rightarrow$ sum of roots $= -\frac{b}{a} = -4$

Example 34. If $\cos 2\theta = 0$, then evaluate $\begin{vmatrix} 0 & \cos\theta & \sin\theta \\ \cos\theta & \sin\theta & 0 \\ \sin\theta & 0 & \cos\theta \end{vmatrix}^2$ (NCERT Exemplar)

Solution. $\begin{vmatrix} 0 & \cos\theta & \sin\theta \\ \cos\theta & \sin\theta & 0 \\ \sin\theta & 0 & \cos\theta \end{vmatrix} = -\cos\theta\begin{vmatrix} \cos\theta & 0 \\ \sin\theta & \cos\theta \end{vmatrix} + \sin\theta\begin{vmatrix} \cos\theta & \sin\theta \\ \sin\theta & 0 \end{vmatrix} = -\cos^3\theta + \sin^3\theta$

Given $\cos 2\theta = 0 \Rightarrow \theta = \frac{\pi}{4}$ so $\Delta = -\cos^3\theta + \sin^3\theta = 0 \Rightarrow \Delta^2 = 0$

Example 35. If $x = -9$ is a root of $\begin{vmatrix} x & 3 & 7 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0$, then find the other two roots. (NCERT Exemplar)

Solution. Given $\begin{vmatrix} x & 3 & 7 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0 \Rightarrow x\begin{vmatrix} x & 2 \\ 6 & x \end{vmatrix} - 3\begin{vmatrix} 2 & 2 \\ 7 & x \end{vmatrix} + 7\begin{vmatrix} 2 & x \\ 7 & 6 \end{vmatrix} = 0 \Rightarrow x^3 - 67x + 126 = 0$

$\because x = -9$ is a root of the given equation $\Rightarrow (x+9)(x^2 - 9x + 14) = 0$

$\therefore$ Other 2 roots are 2 and 7.

Example 36. If A+B+C = 0, then prove that $\begin{vmatrix} 1 & \cos C & \cos B \\ \cos C & 1 & \cos A \\ \cos B & \cos A & 1 \end{vmatrix} = 0$ (NCERT Exemplar)

Solution. $\begin{vmatrix} 1 & \cos C & \cos B \\ \cos C & 1 & \cos A \\ \cos B & \cos A & 1 \end{vmatrix}$ On $R_2 \to R_2 - \cos C\, R_1, R_3 \to R_3 - \cos B\, R_1$

$$= \begin{vmatrix} 1 & \cos C & \cos B \\ 0 & 1-\cos^2 C & \cos A - \cos B \cos C \\ 0 & \cos A - \cos B \cos C & 1-\cos^2 B \end{vmatrix}$$

$$= \begin{vmatrix} \sin^2 C & \cos(B+C) - \cos B \cos C \\ \cos(B+C) - \cos B \cos C & \sin^2 C \end{vmatrix} = \begin{vmatrix} \sin^2 C & -\sin B \sin C \\ -\sin B \sin C & \sin^2 C \end{vmatrix} = 0$$

Example 37. Prove that $\begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & \mathrm{bc}-\mathrm{a}^2 & ca-b^2 \end{vmatrix}$ is divisible by a+b+c and find the quotient. (NCERT Exemplar)

Solution. $\begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & \mathrm{bc}-\mathrm{a}^2 & ca-b^2 \end{vmatrix} = \begin{vmatrix} ab+bc+ca-(a^2+b^2+c^2) & ca-b^2 & ab-c^2 \\ ab+bc+ca-(a^2+b^2+c^2) & ab-c^2 & bc-a^2 \\ ab+bc+ca-(a^2+b^2+c^2) & \mathrm{bc}-\mathrm{a}^2 & ca-b^2 \end{vmatrix}$

$(\,On\ C_1 \to C_1 + C_2 + C_3)$

$= [ab+bc+ca-(a^2+b^2+c^2)] \begin{vmatrix} 1 & ca-b^2 & ab-c^2 \\ 1 & ab-c^2 & bc-a^2 \\ 1 & \mathrm{bc}-\mathrm{a}^2 & ca-b^2 \end{vmatrix}$ (On taking common from C_1)

$= -[a^2+b^2+c^2\text{-}\ ab+bc+ca] \begin{vmatrix} 0 & ca-ab+c^2-b^2 & ab-bc+a^2-c^2 \\ 0 & ab-bc+a^2-c^2 & bc-ca+b^2-a^2 \\ 1 & \mathrm{bc}-\mathrm{a}^2 & ca-b^2 \end{vmatrix}$

(On $R_1 \to R_1 - R_2, R_2 \to R_2 - R_3$)

$= -[a^2+b^2+c^2\text{-}\ (ab+bc+ca)] \begin{vmatrix} ca-ab+c^2-b^2 & ab-bc+a^2-c^2 \\ ab-bc+a^2-c^2 & bc-ca+b^2-a^2 \end{vmatrix}$

$= -[a^2+b^2+c^2\text{-}\ (ab+bc+ca)] \begin{vmatrix} -(a+b+c)(b-c) & -(a+b+c)(c-a) \\ -(a+b+c)(c-a) & -(a+b+c)(a-b) \end{vmatrix}$

$= -[a^2+b^2+c^2\text{-}\ (ab+bc+ca)](a+b+c)^2 \begin{vmatrix} b-c & c-a \\ c-a & a-b \end{vmatrix}$ (On taking $(a+b+c)$ common from C_1 and C_2)

$= -[a^2+b^2+c^2\text{-}\ (ab+bc+ca)](a+b+c)^2(ab+bc+ca-(a^2+b^2+c^2))$

$= [a^2+b^2+c^2\text{-}\ (ab+bc+ca)]^2\,(a+b+c)^2$ which is divisible by $(a+b+c)$ and quotient is

$(a+b+c)[a^2+b^2+c^2\text{-}\ (ab+bc+ca)]^2$

Example 38. **If a+b+c ≠ 0 and $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$, then using properties of determinants,**

prove that a = b = c.(SP14) (NCERT Exemplar)

Solution. $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = \begin{vmatrix} a+b+c & b & c \\ a+b+c & c & a \\ a+b+c & a & b \end{vmatrix}$ $(On\ C_1 \to C_1 + C_2 + C_3)$

= $(a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix}$ **(On taking $(a+b+c)$ common from C_1)**

= $(a+b+c)\begin{vmatrix} 0 & b-c & c-a \\ 0 & c-a & a-b \\ 1 & a & b \end{vmatrix}$ **(On $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$)**

$(a+b+c)\begin{vmatrix} b-c & c-a \\ c-a & a-b \end{vmatrix} = (a+b+c)\,[ab+bc+ca-(a^2+b^2+c^2)]$

= $-(a+b+c)[a^2+b^2+c^2 - (ab+bc+ca)] = -\frac{1}{2}(a+b+c)[(a-b)^2+(b-c)^2+(c-a)^2]$

Given $\Delta = 0 \Rightarrow -\frac{1}{2}(a+b+c)\left[(a-b)^2+(b-c)^2+(c-a)^2\right] = 0$

$\Rightarrow \left[(a-b)^2+(b-c)^2+(c-a)^2\right] = 0$ **(Given a+b+c ≠ 0)**

$\Rightarrow a-b=0, b-c=0, c-a=0 \Rightarrow$ **a = b = c.**

Example 39. If a, b and c are positive and unequal, show that value of the determinant

$\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ **is negative.** (NCERT)

Solution. $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = \begin{vmatrix} a+b+c & b & c \\ a+b+c & c & a \\ a+b+c & a & b \end{vmatrix}$ $(On\ C_1 \to C_1 + C_2 + C_3)$

= $(a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix}$ **(On taking $(a+b+c)$ common from C_1)**

= $(a+b+c)\begin{vmatrix} 0 & b-c & c-a \\ 0 & c-a & a-b \\ 1 & a & b \end{vmatrix}$ **(On $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$)**

$(a+b+c)\begin{vmatrix} b-c & c-a \\ c-a & a-b \end{vmatrix} = (a+b+c)\,[ab+bc+ca-(a^2+b^2+c^2)]$

= $-(a+b+c)[a^2+b^2+c^2 - (ab+bc+ca)] = -\frac{1}{2}(a+b+c)[(a-b)^2+(b-c)^2+(c-a)^2]$

Which is negative as $(a+b+c) > 0$ $(sum\ of\ positive\ numbers)$ and $(a-b)^2+(b-c)^2+(c-a)^2 > 0$ $(a, b\ and\ c\ are\ unequal)$.

Example 40. If a, b and c are real numbers, and $\Delta = \begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0,$

Show that either a + b + c = 0 or a = b = c. (NCERT)

Solution. $\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = \begin{vmatrix} 2(a+b+c) & c+a & a+b \\ 2(a+b+c) & a+b & b+c \\ 2(a+b+c) & b+c & c+a \end{vmatrix}$ ($On\ C_1 \to C_1 + C_2 + C_3$)

$= 2(a+b+c)\begin{vmatrix} 1 & c+a & a+b \\ 1 & a+b & b+c \\ 1 & b+c & c+a \end{vmatrix} = 2(a+b+c)\begin{vmatrix} 0 & c-b & a-c \\ 0 & a-c & b-a \\ 1 & b+c & c+a \end{vmatrix}$ **(On** $R_1 \to R_1 - R_2$ *and* $R_2 \to R_2 - R_3$ **)**

$= 2(a+b+c)\begin{vmatrix} c-b & a-c \\ a-c & b-a \end{vmatrix} = 2(a+b+c)[ab+bc+ca-(a^2+b^2+c^2)]$

$= -2(a+b+c)[0a^2+b^2+c^2-(ab+bc+ca)] = -(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c})[(\boldsymbol{a}-\boldsymbol{b})^2+(\boldsymbol{b}-\boldsymbol{c})^2+(\boldsymbol{c}-\boldsymbol{a})^2]$

Given $\Delta = 0 \Rightarrow -(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c})[(\boldsymbol{a}-\boldsymbol{b})^2+(\boldsymbol{b}-\boldsymbol{c})^2+(\boldsymbol{c}-\boldsymbol{a})^2] = \mathbf{0}$

$\Rightarrow$ Either a + b + c = 0 or $(\boldsymbol{a}-\boldsymbol{b})^2+(\boldsymbol{b}-\boldsymbol{c})^2+(\boldsymbol{c}-\boldsymbol{a})^2 = \mathbf{0}$
$\Rightarrow \boldsymbol{a}-\boldsymbol{b} = \mathbf{0}, \boldsymbol{b}-\boldsymbol{c} = \mathbf{0}, \boldsymbol{c}-\boldsymbol{a} = \mathbf{0} \Rightarrow \boldsymbol{a} = \boldsymbol{b} = \boldsymbol{c}$

Example 41. Show that $\Delta = \begin{vmatrix} (y+z)^2 & xy & zx \\ xy & (x+z)^2 & yz \\ xz & yz & (x+y)^2 \end{vmatrix} = 2xyz(x+y+z)^3$ (NCERT)

Solution. On applying $R_1 \to xR_1, R_2 \to yR_2, R_3 \to zR_3$

$$\Delta = \frac{1}{xyz}\begin{vmatrix} x(y+z)^2 & x^2y & zx^2 \\ xy^2 & y(x+z)^2 & y^2z \\ xz^2 & yz^2 & z(x+y)^2 \end{vmatrix}$$

Taking Common x, y and z from C_1, C_2 *and* C_3 *respectively*

$$= \frac{xyz}{xyz}\begin{vmatrix} (y+z)^2 & x^2 & x^2 \\ y^2 & (x+z)^2 & y^2 \\ z^2 & z^2 & (x+y)^2 \end{vmatrix} = \begin{vmatrix} (y+z)^2 & x^2 & x^2 \\ y^2 & (x+z)^2 & y^2 \\ z^2 & z^2 & (x+y)^2 \end{vmatrix}$$

On applying $C_2 \to C_2 - C_1, C_3 \to C_3 - C_1$

$$= \begin{vmatrix} (y+z)^2 & x^2-(y+z)^2 & x^2-(y+z)^2 \\ y^2 & (x+z)^2-y^2 & 0 \\ z^2 & 0 & (x+y)^2-z^2 \end{vmatrix}$$

Taking Common $(x+y+z)$ from C_2 *and* C_3

$$= (x+y+z)^2\begin{vmatrix} (y+z)^2 & x-y-z & x-y-z \\ y^2 & x+z-y & 0 \\ z^2 & 0 & x+y-z \end{vmatrix}$$

On applying $R_1 \to R_1 - (R_2 + R_3)$ On applying $C_2 \to yC_2, C_3 \to zC_3$

$$= (x+y+z)^2 \begin{vmatrix} 2yz & -2z & -2y \\ y^2 & x+z-y & 0 \\ z^2 & 0 & x+y-z \end{vmatrix} = \frac{(x+y+z)^2}{yz} \begin{vmatrix} 2yz & -2yz & -2yz \\ y^2 & y(x+z-y) & 0 \\ z^2 & 0 & z(x+y-z) \end{vmatrix}$$

On applying $C_2 \rightarrow C_2 + C_1, C_3 \rightarrow C_3 + C_1$

$$= \frac{(x+y+z)^2}{yz} \begin{vmatrix} 2yz & 0 & 0 \\ y^2 & xy+yz & y^2 \\ z^2 & z^2 & zx+yz \end{vmatrix} = \frac{(x+y+z)^2}{yz} X\ 2yz \begin{vmatrix} xy+yz & y^2 \\ z^2 & zx+yz \end{vmatrix}$$

Taking Common y and z from R_1 and R_2 *respectively*

$$= 2(x+y+z)^2 yz \begin{vmatrix} x+z & y \\ z & x+y \end{vmatrix} = 2xyz(x+y+z)^3.$$

20. Solution of Simultaneous Linear Equations AX=B
    $X = A^{-1}B$

21. Consistency of Simultaneous Linear Equations
    Case(i) If $|A| \neq 0$ then Consistent with unique solution
    Case(ii) If $|A| = 0$ then check if (a) adjA X B$\neq$ 0 then InConsistent
    (b) adjA X B=0 then the system has infinitely many solutions or no solution.

➡ **Remarks:** For system of equations with B=0, called homogeneous system of equations are always consistent with trivial solution x=0,y=0,z=0. If |A|=0 then it have infinitely many solution, as in this case adjA X B=0.

Example 42. Two factories decided to award their employees for 3 values (a) adaptable to new techniques,(b) careful and alert in difficult situations and (c) keeping calm in tense situations, at the rate of ₹ x, ₹ y and ₹ z respectively per person. The first factory decided to honour respectively 2,4 and 3 employees with a total prize money of ₹29000. The second factory decided to honour respectively 5,2 and 3 employees with a total prize money of ₹ 30500. If the 3 prizes per person together cost ₹ 9500, then

(i) represent the above situation by a matrix equation and form linear equation using matrix multiplication.
(ii) solve these equations using matrices.
(iii)which values are reflected in this question?(AIC 2013)

Solution. (i) Matrix equation can be given as

$$\begin{bmatrix} 2 & 4 & 3 \\ 5 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 29000 \\ 30500 \\ 9500 \end{bmatrix} \Rightarrow 2x+4y+3z = 29000, 5x+2y+3z = 30500, x+y+z = 9500$$

(ii) Let A= $\begin{bmatrix} 2 & 4 & 3 \\ 5 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$, X= $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and B= $\begin{bmatrix} 29000 \\ 30500 \\ 9500 \end{bmatrix}$ so AX = B⇒ $X = A^{-1}B$

$|A| = \begin{vmatrix} 2 & 4 & 3 \\ 5 & 2 & 3 \\ 1 & 1 & 1 \end{vmatrix} = -1 \neq 0, so\ A^{-1}$ exists.

For Adjoint of A, cofactors are $A_{11} = \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = -1, A_{12} = -\begin{vmatrix} 5 & 3 \\ 1 & 1 \end{vmatrix} = -2, A_{13} = \begin{vmatrix} 5 & 2 \\ 1 & 1 \end{vmatrix} = 3$

$A_{21} = -1, A_{22} = -1\ A_{23} = 2;\ \ A_{31} = 6, A_{32} = 9, A_{33} = -16;$

Now $A^{-1} = \frac{adj\ A}{|A|} = \frac{1}{-1}\begin{bmatrix} -1 & -2 & 3 \\ -1 & -1 & 2 \\ 6 & 9 & -16 \end{bmatrix}^T = \frac{1}{-1}\begin{bmatrix} -1 & -1 & 6 \\ -2 & -1 & 9 \\ 3 & 2 & -16 \end{bmatrix}$

$A^{-1} = \begin{bmatrix} 1 & 1 & -6 \\ 2 & 1 & -9 \\ -3 & -2 & 16 \end{bmatrix} \Rightarrow X = \begin{bmatrix} 1 & 1 & -6 \\ 2 & 1 & -9 \\ -3 & -2 & 16 \end{bmatrix}\begin{bmatrix} 29000 \\ 30500 \\ 9500 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4000 \\ 5000 \\ 3000 \end{bmatrix}$

x= ₹ 4000,y =₹ 5000 and z= ₹ 3000

(iii) Values: (a) adaptable to new techniques,(b) careful and alert in difficult situations and (c) keeping calm in tense situations.

Example 43. In a Legislative assembly election, a political party hired a public relation firm to promote its candidate in 3 ways; telephone, house calls and letters. The no. of contacts of each type in 3 cities A, B, C are (500,1000 and 5000),(3000,1000 and 10000) and (2000,1500 and 4000) respectively. The party paid ₹3700, ₹7200 and ₹4300 in cities A, B and C resp. Find the costs per contact using matrix methods. Keeping in mind the economic condition of the country, which way of promotion is better in your view?

Solution. Let cost of per contact through telephone, house calls and letters are respectively ₹ x , ₹ y and ₹ z. Now matrix equation can be written as $\begin{bmatrix} 500 & 1000 & 5000 \\ 3000 & 1000 & 10000 \\ 2000 & 1500 & 4000 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3700 \\ 7200 \\ 4300 \end{bmatrix}$

Let A= $\begin{bmatrix} 500 & 1000 & 5000 \\ 3000 & 1000 & 10000 \\ 2000 & 1500 & 4000 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 3700 \\ 7200 \\ 4300 \end{bmatrix}$ so AX = B⇒ $X = A^{-1}B$

In $\begin{bmatrix} 500 & 1000 & 5000 \\ 3000 & 1000 & 10000 \\ 2000 & 1500 & 4000 \end{bmatrix}$ take $1000 = 1$ unit and let the new matrix $B = \begin{bmatrix} 0.5 & 1 & 5 \\ 3 & 1 & 10 \\ 2 & 1.5 & 4 \end{bmatrix}$

$|B| = \begin{vmatrix} 0.5 & 1 & 5 \\ 3 & 1 & 10 \\ 2 & 1.5 & 4 \end{vmatrix} = 0.5\begin{vmatrix} 1 & 10 \\ 1.5 & 4 \end{vmatrix} - 1\begin{vmatrix} 3 & 10 \\ 2 & 4 \end{vmatrix} + 5\begin{vmatrix} 3 & 1 \\ 2 & 1.5 \end{vmatrix} = -5.5 + 8 + 12.5 = 15 \neq 0$

So B^{-1} will exist.

For Adjoint of B, cofactors are

$$A_{11}=\begin{vmatrix}1 & 10\\ 1.5 & 4\end{vmatrix}=-11, A_{12}=-\begin{vmatrix}3 & 10\\ 2 & 4\end{vmatrix}=-8, A_{13}=\begin{vmatrix}3 & 1\\ 2 & 1.5\end{vmatrix}=12.5$$

$$A_{21}=-3.5, A_{22}=-8\ A_{23}=1.25;\quad A_{31}=5, A_{32}=10, A_{33}=-2.5;$$

Now $B^{-1}=\frac{adj\,B}{|B|}=\frac{1}{15}\begin{bmatrix}-11 & -8 & 12.5\\ -3.5 & -8 & 1.25\\ 5 & 10 & -2.5\end{bmatrix}^T=\frac{1}{15}\begin{bmatrix}-11 & -3.5 & 12.5\\ -8 & -8 & 1.25\\ 12.5 & 10 & -2.5\end{bmatrix}$

$$\Rightarrow A^{-1}=\frac{1}{15000000}\begin{bmatrix}-11000 & -3500 & 12500\\ -8000 & -8000 & 12500\\ 12500 & 10000 & -2500\end{bmatrix}$$

$$X=\frac{1}{15000000}\begin{bmatrix}-11000 & -3500 & 12500\\ -8000 & -8000 & 12500\\ 12500 & 10000 & -2500\end{bmatrix}\begin{bmatrix}3700\\ 7200\\ 4300\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\frac{1}{15000000}\begin{bmatrix}6000000\\ 15000000\\ 7500000\end{bmatrix}\Rightarrow\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0.4\\ 1\\ 0.5\end{bmatrix}$$

x= ₹ 0.4,y =₹ 1 and z= ₹ 0.5 ; Telephone is better as it is cheaper.

Example 44. Find $\begin{pmatrix}-4 & 4 & 4\\ -7 & 1 & 3\\ 5 & -3 & -1\end{pmatrix}\begin{pmatrix}1 & -1 & 1\\ 1 & -2 & -2\\ 2 & 1 & 3\end{pmatrix}$ use this to solve the following system of equations:

x-y+z=4
x-2y-2z=9 (DelhiC 2010), (DelhiC 2012)
2x+y+3z=1

Solution. $\begin{pmatrix}-4 & 4 & 4\\ -7 & 1 & 3\\ 5 & -3 & -1\end{pmatrix}\begin{pmatrix}1 & -1 & 1\\ 1 & -2 & -2\\ 2 & 1 & 3\end{pmatrix}=\begin{pmatrix}8 & 0 & 0\\ 0 & 8 & 0\\ 0 & 0 & 8\end{pmatrix}=8\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}=8I$

Hence $\begin{pmatrix}1 & -1 & 1\\ 1 & -2 & -2\\ 2 & 1 & 3\end{pmatrix}^{-1}=\frac{1}{8}\begin{pmatrix}-4 & 4 & 4\\ -7 & 1 & 3\\ 5 & -3 & -1\end{pmatrix}$

Given system of equation in matrix form is given by $\begin{pmatrix}1 & -1 & 1\\ 1 & -2 & -2\\ 2 & 1 & 3\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}4\\ 9\\ 1\end{pmatrix}$

Hence $\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}1 & -1 & 1\\ 1 & -2 & -2\\ 2 & 1 & 3\end{pmatrix}^{-1}\begin{pmatrix}4\\ 9\\ 1\end{pmatrix}=\frac{1}{8}\begin{pmatrix}-4 & 4 & 4\\ -7 & 1 & 3\\ 5 & -3 & -1\end{pmatrix}\begin{pmatrix}4\\ 9\\ 1\end{pmatrix}=\frac{1}{8}\begin{pmatrix}-16+36+4\\ -28+9+3\\ 20-27-1\end{pmatrix}=\frac{1}{8}\begin{pmatrix}24\\ -16\\ -8\end{pmatrix}$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix} \Rightarrow x = 3, y = -2, z = -1.$$

Important Problems for Practice

For 1 mark

1. Construct a matrix of order 2X2 if $a_{ij}=(i-j)^2$. $\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)$
2. For a 2 X 2 matrix A=[a_{ij}], whose elements are given by $a_{ij} = \frac{(i+2j)^2}{8}$, write the value of a_{21}. (2)
3. **Construct a matrix of order 2X2 whose elements are given by**
   $A_{ij}=\begin{cases} \boldsymbol{i-j} & \boldsymbol{if\ i \geq j} \\ \boldsymbol{i+j} & \boldsymbol{if\ i < j} \end{cases}$ $\left(\begin{bmatrix} 0 & 3 \\ 1 & 0 \end{bmatrix}\right)$
4. Evaluate x and y if
   $\begin{pmatrix} 3+x & 2 \\ 7 & y+1 \end{pmatrix} = \begin{pmatrix} 7 & 2 \\ 7 & 0 \end{pmatrix}$ $(4, -1)$
5. If A is a matrix of order 3X4 and B is a matrix of order 4X3, find the order of the matrix AB. (DelhiC 2010) $(3X3)$
6. If A = $\begin{bmatrix} 21 & 4 \\ 41 & 5 \end{bmatrix}$, B = $\begin{bmatrix} 3 & 6 \\ 4 & 1 \\ 0 & 9 \end{bmatrix}$ write the order of AB'. $(2X3)$
7. Find X if Y = $\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$ and 2X+Y = $\begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}$. $\left(\begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix}\right)$
8. If 3A-B = $\begin{bmatrix} 5 & 0 \\ 1 & 1 \end{bmatrix}$ and B = $\begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$, then find the matrix A. (DelhiC 2012) $\left(\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}\right)$
9. If A=$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $find\ A + A'$. (AIC10) $\left(\begin{bmatrix} 2 & 5 \\ 5 & 8 \end{bmatrix}\right)$
10. If $\begin{pmatrix} 2x+y & 3y \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 6 & 4 \end{pmatrix}'$, then find x.(AIC10) (2)
11. If $2\begin{pmatrix} 3 & 4 \\ 5 & x \end{pmatrix} + \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 7 & 0 \\ 10 & 5 \end{pmatrix}$, $find\ (x-y)$.(Delhi 2014) (8)
12. Solve the following matrix equation for x:
    $[x \quad 1]\begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix} = 0$ (Delhi 2014) (2)

13. If $\begin{pmatrix} a+b & 2 \\ 5 & b \end{pmatrix} = \begin{pmatrix} 6 & 5 \\ 2 & 2 \end{pmatrix}, then\ find\ a.$(DelhiC 2010) (4)

14. If $\begin{pmatrix} x-y & 2y \\ 2y+z & x+y \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 9 & 5 \end{pmatrix}$, then write the value of x+y+z. (AIC 2013) (10)

15. If $\begin{pmatrix} x+y & 1 \\ 2y & 5 \end{pmatrix} = \begin{pmatrix} 7 & 1 \\ 4 & 5 \end{pmatrix}, then\ find\ x.$(DelhiC 2010) (5)

16. If $\begin{bmatrix} x & x-y \\ 2x+y & 7 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 8 & 7 \end{bmatrix}$, then find the value of y. (DelhiC 2011) (2)

17. If $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} 7 & 11 \\ k & 23 \end{pmatrix}, then\ write\ the\ value\ of\ k.$(Delhi 2010) (17)

18. From the following matrix equation, find the value of x:
$\begin{pmatrix} x+y & 4 \\ -5 & 3y \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ -5 & 6 \end{pmatrix}$(Foreign10) (1)

19. From the following matrix equation, find the value of x:
$\begin{pmatrix} 1 & 3 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} x \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$(Foreign10) (-1)

20. If $\begin{pmatrix} 3 & 4 \\ 2 & x \end{pmatrix}\begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} 19 \\ 15 \end{pmatrix}$, find the value of x.(Foreign10) (5)

21. Solve for x and y, if $\begin{pmatrix} 2x & 4 \\ -2 & y+x \end{pmatrix} = \begin{pmatrix} 13 & 4 \\ -2 & 5 \end{pmatrix}$ $\left(\frac{13}{2}, -\frac{3}{2}\right)$

22. If $\begin{bmatrix} 2x+1 & 2y \\ 0 & y^2+1 \end{bmatrix} = \begin{bmatrix} x+3 & 10 \\ 0 & 26 \end{bmatrix}$, write the value of (x+y). (AIC 2012) (7)

23. If $\begin{bmatrix} x & x-y \\ 2x+y & 7 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 8 & 7 \end{bmatrix}$, then find the value of y. (DelhiC 2011) (2)

24. **For what value of k, the matrix** $\begin{pmatrix} \mathbf{2k+3} & \mathbf{4} & \mathbf{5} \\ \mathbf{-4} & \mathbf{0} & \mathbf{-6} \\ \mathbf{-5} & \mathbf{6} & \mathbf{-2k-3} \end{pmatrix}$ **is skew- symmetric? (SP14)** $\left(-\frac{3}{2}\right)$

25. **For what value of a and b the matrix** $\begin{pmatrix} \mathbf{0} & \mathbf{3} & \mathbf{5} \\ \boldsymbol{b} & \mathbf{0} & \mathbf{7} \\ \mathbf{-5} & \mathbf{-7} & \boldsymbol{a} \end{pmatrix}$ **is skew -symmetric? (SP14)** $(0, -3)$

26. Find x, if $\begin{vmatrix} 2 & 3 \\ 5 & 5 \end{vmatrix} = \begin{vmatrix} x & 3 \\ 3x & 5 \end{vmatrix}$ $\left(\frac{5}{4}\right)$

27. **If $\begin{vmatrix} \boldsymbol{sin\alpha} & \boldsymbol{cos\beta} \\ \boldsymbol{cos\alpha} & \boldsymbol{sin\beta} \end{vmatrix} = \frac{1}{2}$, where α, β are acute angles, then write the value of α+β. (SP14)** $\left(\frac{2\pi}{3}\right)$

28. If $\begin{vmatrix} 2x & x+3 \\ 2(x+1) & x+1 \end{vmatrix} = \begin{vmatrix} 1 & 5 \\ 3 & 3 \end{vmatrix}$, then write the value of x.(DelhiC 2013) (1)

29. If $\begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & -2 \\ 7 & 3 \end{vmatrix}$, then write the value of x.(Delhi 2014) (± 6)

30. What positive value of x makes the following pair of determinant equal?

$\begin{vmatrix} 2x & 3 \\ 5 & x \end{vmatrix}$, $\begin{vmatrix} 16 & 3 \\ 5 & 2 \end{vmatrix}$(AI 2010) (± 4)

31. What is the value of the determinant $\begin{vmatrix} 0 & 2 & 0 \\ 2 & 3 & 4 \\ 4 & 5 & 6 \end{vmatrix}$? (Delhi 2010) (8)

32. What is the value of the determinant $\begin{vmatrix} 4 & a & b+c \\ 4 & b & c+a \\ 4 & c & a+b \end{vmatrix}$? (Foreign10) (0)

33. For what value of x, the given matrix A= $\begin{pmatrix} 3-2x & x+1 \\ 2 & 4 \end{pmatrix}$ *is a singular matrix*?.(AIC 2013) (1)

34. For what value of x, the matrix $\begin{bmatrix} 1+x & 7 \\ 3-x & 8 \end{bmatrix}$ is a singular matrix? (AIC 2012) $\left(\frac{13}{15}\right)$

35. For what value of x is the matrix $\begin{bmatrix} 2x & 4 \\ x+2 & 3 \end{bmatrix}$ singular? (DelhiC 2011) (4)

36. For what value of x, the given matrix A= $\begin{pmatrix} 6-x & 4 \\ 3-x & 1 \end{pmatrix}$ *is a singular matrix*?. (DelhiC 2011) $\left(-\frac{2}{5}\right)$

37. For what value of x, the given matrix A= $\begin{pmatrix} 2x+4 & 4 \\ x+5 & 3 \end{pmatrix}$ *is a singular matrix*?. (AIC 2011) (4)

38. **For what value of x is the matrix $\begin{bmatrix} \boldsymbol{2(x+1)} & \boldsymbol{2x} \\ \boldsymbol{x} & \boldsymbol{x-2} \end{bmatrix}$ singular? (AIC 2011)** (-2)

39. For what value of k, the matrix $\begin{bmatrix} k & 2 \\ 2 & 5 \end{bmatrix}$ has no inverse ? $\left(\frac{4}{5}\right)$

40. **If A = $\begin{pmatrix} cos\alpha & -sin\alpha \\ sin\alpha & cos\alpha \end{pmatrix}$, then for what value of α is A is an identity matrix? (Delhi 2010)** (0^0)

41. Minor of element of 2nd row and 3rd column (a_{23}) in $\begin{pmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{pmatrix}$(Delhi 2010) (13)

42. Write a square matrix of order 2, which is both symmetric and skew symmetric. (Foreign10) $\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right)$

43. If A =$\begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$, then find the value of 3|A|. (AIC 2011) (6)

44. If matrix A = $\begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$, find A^{15}. $\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right)$

45. If $|A^3|$=125, evaluate $|A^2|$-4|A|+7 where A is a square matrix of order 3X3. (12)

46. If A = $\begin{bmatrix} 2 & 3 & 3 \\ 0 & 2 & 0 \\ 5 & 5 & 7 \end{bmatrix}$ then evaluate |A.(adjA)|. (-8)

47. The value of the determinant of a matrix A of order 3X3 is 4. Find the value of |5A|. (DelhiC 2012) (500)

48. If A is a matrix of order 3X3 is such that |A|=4. Find the value of |2A|. (DelhiC 2011) (32)

49. **If A is a square matrix of order 3 and |3A|=K|A|, then write the value of K. (Delhi 2010)** (27)

50. Find order of matrix such that |2A| = 8 |A|. $(3X3)$

51. Write the adjoint of the matrix $\begin{pmatrix} 2 & -1 \\ 4 & 3 \end{pmatrix}$(AI 2010) $\left(\begin{bmatrix} 3 & 1 \\ -4 & 2 \end{bmatrix}\right)$

52. If A = $\begin{pmatrix} 3 & 1 \\ 2 & -3 \end{pmatrix}$, *then find* $|adjA|$.(DelhiC 2010) (-11)

53. Find A.(adj A) when A = $\begin{pmatrix} 3 & 4 \\ -3 & 0 \end{pmatrix}$ $\left(\begin{bmatrix} 144 & 0 \\ 0 & 144 \end{bmatrix}\right)$

54. **If A = $\begin{pmatrix} 4 & 6 \\ 7 & 5 \end{pmatrix}$, then what is A.(adj A)? (SP14)** $\left(\begin{bmatrix} 484 & 0 \\ 0 & 484 \end{bmatrix}\right)$

55. **If |A| = 12, where A is a 3X3 matrix, find |A.adj A|.** (1728)

56. **A is a square matrix of order 3 and $|A|$ = 7. Write the value of |adj A|.** (49)

57. If |A| = 2, where A is a 2X2 matrix, find |adj A|. (AIC10) (2)

58. **If A is a square matrix of order 3 such that |adj A|= 64, find |A|. (DelhiC 2013)** (± 8)

59. A is a square matrix of order 3 and $|A|$ =7. Write the value of |adj A|. (AI 2010) (49)
60. If A is an invertible matrix of order 3 and |A|= 5 find the value of |adj A|. (AIC 2011) (25)

For 4 marks

1. Find the equation of the line joining A(1,3) and B(0,0) using determinants and find the value of k if D(k,0) is a point such that area of triangle ABD is 3 square units. (AIC 2013) (NCERT) $(3x - y = 0, k = 2)$
2. For the following matrices A and B, verify that (AB)'=B'A'
   A= $\begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix}$, B= $\begin{pmatrix} -1 & 2 & 1 \end{pmatrix}$ (AI 2010)
3. Express the following matrix as the sum of a symmetric matrix and skew symmetric matrix and verify your result: $\begin{pmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{pmatrix}$ $\left(\begin{bmatrix} 3 & \frac{1}{2} & -\frac{5}{2} \\ \frac{1}{2} & -2 & -2 \\ -\frac{5}{2} & -2 & 2 \end{bmatrix}, \begin{bmatrix} 0 & -\frac{5}{2} & -\frac{3}{2} \\ \frac{5}{2} & 0 & -3 \\ \frac{3}{2} & 3 & 0 \end{bmatrix}\right)$
4. Express the matrix A = $\begin{bmatrix} 2 & 0 & -3 \\ 4 & 3 & 1 \\ -5 & 7 & 2 \end{bmatrix}$ as sum of symmetric and skew symmetric matrix.
   $\left(\begin{bmatrix} 2 & 2 & -4 \\ 2 & 3 & 4 \\ -4 & 4 & 2 \end{bmatrix}, \begin{bmatrix} 0 & -2 & 1 \\ 2 & 0 & 3 \\ -1 & 3 & 0 \end{bmatrix}\right)$
5. If A = $\begin{pmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{pmatrix}$, then find the value of $A^2 - 3A + 2I$. (AI 2010) $\left(\begin{bmatrix} 1 & -1 & -1 \\ 3 & -3 & -4 \\ -3 & 2 & -4 \end{bmatrix}\right)$
6. **If A = $\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$. Show that A^2 -5A +7 I =0 and hence find A^{-1}.** $\left(\begin{bmatrix} \frac{2}{7} & -\frac{1}{7} \\ \frac{1}{7} & \frac{3}{7} \end{bmatrix}\right)$
7. Find the inverse of A=$\begin{pmatrix} 3 & -1 \\ -4 & 1 \end{pmatrix}$ using elementary transformations.(Foreign10) $\left(\begin{bmatrix} -1 & -1 \\ -4 & -3 \end{bmatrix}\right)$
8. Find the inverse of the following matrix, using elementary transformation :
   A=$\begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix}$(Foreign10) $\left(\begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}\right)$

9. Find the inverse of the following matrix, using elementary transformation:
A=$\begin{pmatrix}6 & 5\\5 & 4\end{pmatrix}$(Foreign10) $\left(\begin{bmatrix}-4 & 5\\5 & -6\end{bmatrix}\right)$

10. If A = $\begin{bmatrix}7 & 1\\4 & -3\end{bmatrix}$ find A^{-1} by using elementary transformation. $\left(\begin{bmatrix}\frac{3}{25} & \frac{1}{25}\\\frac{4}{25} & -\frac{7}{25}\end{bmatrix}\right)$

11. Using properties of determinants, prove the following :

$\begin{vmatrix}1 & a & a^3\\1 & b & b^3\\1 & c & c^3\end{vmatrix}$=(a-b)(b-c)(c-a)(a+b+c) (DelhiC 2013)

12. Using properties of determinants, prove the following :

$\begin{vmatrix}1 & 1 & 1\\a & b & c\\a^3 & b^3 & c^3\end{vmatrix}$ =(a-b)(b-c)(c-a)(a+b+c) (DelhiC 2011)

13. Using properties of determinants, prove the following :

$\begin{vmatrix}x & y & z\\x^2 & y^2 & z^2\\x^3 & y^3 & z^3\end{vmatrix}$ =xyz(x-y) (y-z) (z-x) (DelhiC 2010)

14. Using properties of determinants, prove the following:
$\begin{vmatrix}\alpha & \beta & \gamma\\\alpha^2 & \beta^2 & \gamma^2\\\beta+\gamma & \alpha+\gamma & \alpha+\beta\end{vmatrix}$=(α-β)(β-γ)(γ-α)(α+β+γ). (DelhiC 2010), (DelhiC 2012)

15. **Using properties of determinants, prove the following :**

$\begin{vmatrix}(b+c)^2 & a^2 & a^2\\b^2 & (c+a)^2 & b^2\\c^2 & c^2 & (a+b)^2\end{vmatrix}$ **=2abc(a+b+c)^3 (AIC10)**

16. Prove the following, using properties of determinants:
$\begin{vmatrix}a+b+2c & a & b\\c & b+c+2a & b\\c & a & c+a+2b\end{vmatrix}$ = 2(a+b+c)^3 (Foreign10), (DelhiC 2012), (Delhi 2014)

17. Prove the following, using properties of determinants:
$\begin{vmatrix}a+b+2c & b & a\\b & b+c+2a & c\\a & c & c+a+2b\end{vmatrix} = 6(a+b+c)(ab+bc+ca)$

18. Prove the following, using properties of determinants:

$$\begin{vmatrix} 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \\ x-y-z & 2x & 2x \end{vmatrix} = (x+y+z)^3$$ (Delhi 2014)

19. Prove the following, using properties of determinants:
$$\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 2(3abc-a^3-b^3-c^3)$$ (Foreign10)
20. Using properties of determinants, prove the following :
$$\begin{vmatrix} b+c & a-b & a \\ c+a & b-c & b \\ a+b & c-a & c \end{vmatrix} = 3abc- a^3- b^3- c^3$$ (AIC 2012)

21. Using properties of determinants, prove the following :
$$\begin{vmatrix} b+c & a-b & a \\ c+a & b-c & b \\ a+b & c-a & c \end{vmatrix} = 3abc- a^3- b^3- c^3$$ (AIC 2012)
22. **Using properties of determinants prove the following:**

$$\begin{vmatrix} \boldsymbol{a^2} & \boldsymbol{a^2-(b-c)^2} & \boldsymbol{bc} \\ \boldsymbol{b^2} & \boldsymbol{b^2-(c-a)^2} & \boldsymbol{ca} \\ \boldsymbol{c^2} & \boldsymbol{c^2-(a-b)^2} & \boldsymbol{ab} \end{vmatrix} = \mathbf{(a-b)(b-c)(c-a)(a+b+c)(a^2+b^2+c^2)}.$$ **(AIC 2012)**

23. **Using properties of determinants prove the following:**

$$\begin{vmatrix} \boldsymbol{a} & \boldsymbol{b-c} & \boldsymbol{c+b} \\ \boldsymbol{a+c} & \boldsymbol{b} & \boldsymbol{c-a} \\ \boldsymbol{a-b} & \boldsymbol{b+a} & \boldsymbol{c} \end{vmatrix} = \mathbf{(a+b+c)(a^2+b^2+c^2)}.$$ **(AIC 2012)**

24. Using properties of determinants prove the following:

$$\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca).$$ (DelhiC 2011)

25. Using properties of determinants, prove the following :
$$\begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix} = 4a^2b^2c^2$$ (AIC 2011,Delhi 2011)

26. Using properties of determinants prove the following:
$$\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix} = 1+a^2+b^2+c^2$$ (AIC 2011), (Foreign 13)
27. Using properties of determinant, evaluate $\begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix}$ $((a-1)^3)$

28. Using properties of determinant prove that

$$\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2$$

29. Using properties of determinant prove that

$$\begin{vmatrix} -bc & b^2+bc & c^2+bc \\ a^2+ac & -ac & c^2+ac \\ a^2+ab & b^2+ba & -ab \end{vmatrix} = (ab+bc+ac)^3$$

30. Using properties of determinant prove that

$$\begin{vmatrix} x^2+1 & xy & xz \\ yx & y^2+1 & yz \\ zx & zy & z^2+1 \end{vmatrix} = 1+x^2+y^2+z^2.$$ (Delhi 2014)

31. **Show that x = 2 is a root of the equation formed by the following** $\begin{vmatrix} x & -6 & -1 \\ 2 & -3x & x-3 \\ -3 & 2x & x+2 \end{vmatrix} = 0$**, and hence solve the equation.**

32. **If x ,y and z are unequal and** $\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0$**, prove that 1+xyz = 0.**

For 6 marks

1. **Using properties of determinants. Show the following:**

$$\begin{vmatrix} (b+c)^2 & ab & ca \\ ab & (a+c)^2 & bc \\ ac & bc & (a+b)^2 \end{vmatrix} = 2abc(a+b+c)^3$$ **(Delhi 2010)**

2. Using properties of determinants, prove the following:

$$\begin{vmatrix} x & x^2 & 1+px^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz^3 \end{vmatrix} = (1+pxyz)(x-y)(y-z)(z-x)$$ (AI 2010)

3. **If a, b, c are positive and unequal, show that the following determinants is negative:**

$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ **(AI 2010)**

4. Find the inverse of the following matrix using elementary operations:

A= $\begin{pmatrix} 0 & 1 & 3 \\ 1 & 2 & 1 \\ 2 & 3 & 1 \end{pmatrix}$ $\left(A^{-1} = \begin{bmatrix} \frac{1}{2} & -4 & \frac{5}{2} \\ -\frac{1}{2} & 3 & -\frac{3}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix} \right)$

5. Using matrices solve the following system of equations:

x+2y-3z=-4

2x+3y+2z=2

3x-3y-4z=11(AI 2010)

$\left(\frac{1}{67} \begin{bmatrix} -6 & 17 & 13 \\ 14 & 5 & -8 \\ -15 & 9 & -1 \end{bmatrix} \quad x = 3, y = -2, z = 1 \right)$

6. Using matrices, solve the following system of equations:

3x-2y+3z=-1; 2x+y-z=6; 4x-3y+2z=5 (Foreign10)

$\left(A^{-1} = \frac{1}{-17} \begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix} \quad x = 2, y = -1, z = -3 \right)$

7. Solve using matrix method, 3x+4y+7z = 14, 2x-y+3z = 4, x+2y-3z = 0. $(x = 1, y = 1, z = 1)$

8. Solve using matrix method x-y+2z = 1, 2y-3z = 1, 3x-2y+4z = 2. $(x = 0, y = 5, z = 3)$

9. **Using matrices solve the following system of equations:**

$\frac{1}{x} - \frac{1}{y} + \frac{1}{z} = 4$

$\frac{2}{x} + \frac{1}{y} - \frac{3}{z} = 0$

$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 2 \quad x \neq 0, y \neq 0, z \neq 0.$

$\left(A^{-1} = \frac{1}{10} \begin{pmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{pmatrix}; x = \frac{1}{2}, y = -1, z = 1 \right)$

10. **Two schools A and B want to award their selected teachers on the values of honesty, hard work and regularity. The school A wants to award ₹ x, ₹ y and ₹ z each for the 3 respective values to 3,2 and 1 teachers with a total award money of ₹ 1.28 lakhs. school B wants to spend ₹ 1.54 lakhs to award its 4,1 and 3 teachers on the respective vales(by giving the same award money for the 3 values as before). If the total amount of award for one prize on each value is ₹ 57000,using matrix method find the award money for each value.(SP14)**

($A^{-1} = \frac{-1}{5}\begin{pmatrix}-2 & -1 & 5\\ -1 & 2 & -5\\ 3 & -1 & -5\end{pmatrix}$ Award money for honesty ₹ 25000, Award money for hard work ₹ 21000 and Award money for regularity ₹ 11000**)**

11. **Two schools P and Q want to award their selected students on the values of Discipline, Politeness and Punctuality. The school P wants to award ₹ x, ₹ y and ₹ z each for the 3 respective values to its 3,2 and 1 students with a total award money of ₹ 1000. School Q wants to spend ₹ 1,500 to award its 4, 1 and 3 students on the respective vales (by giving the same award money for the 3 values as before). If the total amount of award for one prize on each value is ₹ 600,using matrix method find the award money for each value.**
Apart from the above three values suggest one more value for awards. (Delhi 2014)

($A^{-1} = \frac{-1}{5}\begin{pmatrix}-2 & -1 & 5\\ -1 & 2 & -5\\ 3 & -1 & -5\end{pmatrix}$ Award money for Discipline ₹ 100, Award money for politeness ₹ 200 and Award money for punctuality ₹ 300 **)**

12. Two schools P and Q want to award their selected students on the values of Tolerance, Kindness and Leadership. The school P wants to award ₹ x, ₹ y and ₹ z each for the 3 respective values to its 3,2 and 1 students with a total award money of ₹ 2,200. School Q wants to spend ₹ 3,100 to award its 4, 1 and 3 students on the respective values (by giving the same award money for the 3 values as school P). If the total amount of award for one prize on each value is ₹ 1,200,using matrix method find the award money for each value.
Apart from the above three values suggest one more value for awards. (Foreign14)

($A^{-1} = \frac{1}{-5}\begin{bmatrix}-2 & -1 & 5\\ -1 & 2 & -5\\ 3 & -1 & -5\end{bmatrix}; x = ₹\,300, y = ₹\,400, y = ₹\,500$; one more value like honesty, punctuality etc may be awarded.)

13. Two institutions decided to award their employees for 3 values of resourcefulness, competence and determination in form of prizes at the rate of ₹ x, ₹ y and ₹ z respectively per person. The first institution decided to award respectively 4,3 and 2 employees with a total prize money of ₹ 37,000 and the 2nd institution decided to award respectively 5,3 and 4 employees with a total prize money of ₹ 47,000. If all the 3 prizes per person together amount to ₹ 12,000, then using matrix method find the value of x, y and z. What values are described in this question? (DelhiC 2013)

(4x+3y+2z=37000,5x+3y+4z=47000,x+y+z=12000, $A^{-1} = \frac{-1}{5}\begin{pmatrix}-1 & -1 & 6\\ -1 & 2 & -6\\ 2 & -1 & -3\end{pmatrix}$ Award money for Resourcefulness= ₹ 4000, Award money for Competence =₹ 5000 and Award money for Determination= ₹ 3000 Values: Resourcefulness, Competence, Determination)

14. **Three shopkeepers A, B, C are using polythene, handmade bags and n/p envelop as carry bags. It is found that the shopkeepers A,B,C are using (20,30,40),(30,40,20),(40,20,300 polythene , handmade bags and n/p envelops respectively. The shopkeepers A, B, C spent ₹250, ₹220 and ₹200 on these carry bags respectively. Find the cost of each carry bags using matrices. Keeping in mind the social and environment conditions, which shopkeeper is better? And why?**

$(x = ₹1, y = ₹5, z = ₹2;$ A is *better for environment and B for social conditions.*$)$

15. There are 3 families. First family consists of 2 male members, 4 female members and 3 children. Second family consists of 3 male members, 3 female members and 2 children. Third family consists of male members,2 female members and 5 children. Male members earn ₹500 per day and spend ₹ 300 per day. Female members earns ₹ 400 per day and spend ₹ 250 per day child members spends ₹ 40 per day. Find the money each family saves per day using matrices? What is the necessity of saving in the family?
   $(x = ₹880, y = ₹970, z = ₹500;$ In case of emegency savings help us.)

16. **For keeping fit X people believes in morning walk, Y people believes in Yoga and Z people join Gym. Total no. of people is 70. Further 20%, 30% and 40% people are suffering from any disease who believes in morning walk yoga and Gym resp. Total no. of such people is 21. If morning walk cost ₹ 0 Yoga cost ₹ 500/month and Gym cost ₹ 400/month and total expenditure is ₹ 23000.**

   (i) **Formulate a matrix problem.** $(x + y + z = 70, 2x + 3y + 4z = 210, 5y + 4z = 230)$

   (ii) **Calculate the no. of each type of people.** $(x = ₹20, y = ₹30, z = ₹20;)$

   Why exercise is important for health? (Exercise keeps fit and healthy to a person)

17. If A = $\begin{pmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{pmatrix}$ $and\ B = \begin{pmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{pmatrix}$, find AB use this to solve $the\ following\ system\ of\ equations$

   x-y=3
   2x+3Y+4z=17 $(6I,\ x = 2, y = -1, z = 4)$
   Y+2z=7 (AIC10), (AIC 2012)

18. If A= $\begin{pmatrix} 1 & -2 & 0 \\ 2 & 1 & 3 \\ 0 & -2 & 1 \end{pmatrix}$ $and\ B = \begin{pmatrix} 7 & 2 & -6 \\ -2 & 1 & -3 \\ -4 & 2 & 5 \end{pmatrix}$ find AB hence solve the following system of equations:
   x-2y=10 $(AB = 11I, x = 4, y = -3, z = 1)$
   2x+y+3z=8 (DelhiC 2011)
   -2y+z=7

19. If A^T= $\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$ $and\ B^T = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{pmatrix}$ find AB hence solve the following system of equations:
   2x-y+z= -1
   -x+2y-z= 4 $(AB = 4I, x = 1, y = 2, z = -1)$
   x-y+2z= -3

20. If A = $\begin{pmatrix} 3 & 2 & 1 \\ 4 & -1 & 2 \\ 7 & 3 & -3 \end{pmatrix}$, *then find* A^{-1}. *Hence solve the following system of equations* :

3x+2y+z=6

4x-y+2z=5 $\left(A^{-1} = \frac{1}{62}\begin{bmatrix} -3 & 9 & 5 \\ 26 & -16 & -2 \\ 19 & 5 & -11 \end{bmatrix},\ x = 1, y = 1, z = 1 \right)$

7x+3y-3z=7 (DelhiC 2010)

21. If A = $\begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 1 \\ 1 & -3 & 1 \end{pmatrix}$, *then find* A^{-1}. *Hence solve the following system of equations* :

x+2y+z=4

-x+y+z=0 $\left(A^{-1} = \frac{1}{10}\begin{bmatrix} 4 & -5 & 1 \\ 2 & 0 & -2 \\ 2 & 5 & 3 \end{bmatrix}\ x = 2, y = 0, z = 2 \right)$

x-3y+z=4 (DelhiC 2012)

22. If A = $\begin{pmatrix} 3 & 2 & 1 \\ 4 & -1 & 2 \\ 7 & 3 & -3 \end{pmatrix}$, *then find* A^{-1}. *Hence solve the following system of equations* :

3x+4y+7z=0

2x-y+3z=-2 $\left(A^{-1} = \frac{1}{62}\begin{bmatrix} -3 & 9 & 5 \\ 26 & -16 & -2 \\ 19 & 5 & -11 \end{bmatrix},\ x = 1, y = 1, z = -1 \right)$

x+2y-3z=6

23. If A = $\begin{pmatrix} 8 & -4 & 1 \\ 10 & 0 & 6 \\ 8 & 1 & 6 \end{pmatrix}$, find A^{-1}. Hence solve the following system of equations:

8x+4y+z=5

10x+6z=4 $\left(A^{-1} = \frac{1}{10}\begin{bmatrix} -6 & 25 & -24 \\ -12 & 40 & -38 \\ 10 & -40 & 40 \end{bmatrix}, x = 1, y = \frac{1}{2}, z = -1 \right)$

8x+y+6z= $\frac{5}{2}$ (AIC10)

24. If A= $\begin{pmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{pmatrix}$, find A^{-1}. Hence solve the following system of equations:

2x-3y+5z=16

3x+2y-4z=-4 $\left(A^{-1} = \frac{1}{-1}\begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}\ x = 2, y = 1, z = 3 \right)$

x+y-2z=-3 (Foreign10), (AIC 2012)

25. If A= $\begin{bmatrix} 2 & 1 & 3 \\ 1 & 3 & -1 \\ -2 & 1 & 1 \end{bmatrix}$, find A^{-1}. Using A^{-1}, solve the following system of equations:

2x+y+3z=9; x+3y –z=2; -2x+y +z=7. (Foreign10)

$\left(A^{-1} = \frac{1}{30}\begin{bmatrix} 4 & 2 & -10 \\ 1 & 8 & 5 \\ 7 & -4 & 5 \end{bmatrix}\ x = -1, y = 2, z = 3 \right)$

26. If A = $\begin{bmatrix}1 & 2 & -3\\ 2 & 3 & 2\\ 3 & -3 & -4\end{bmatrix}$, find A^{-1}. Hence solve the system of equations:

X + 2y - 3z = -4, 2x + 3y + 2z = 2, 3x - 3y – 4z = 11. (DelhiC 2012)

$\left(A^{-1} = \frac{1}{67}\begin{bmatrix}-6 & 17 & 13\\ 14 & 5 & -8\\ -15 & 9 & -1\end{bmatrix}, x = 3, y = -2, z = 1\right)$

27. If A = $\begin{bmatrix}1 & 2 & 5\\ 1 & -1 & -1\\ 2 & 3 & -1\end{bmatrix}$, find A^{-1}. Hence solve the system of equations:

X + 2y +5z = 10, x -y -z = -2, 2x +3y – z = -11. (AIC 2012)

$\left(A^{-1} = \frac{1}{27}\begin{bmatrix}4 & 17 & 3\\ -1 & -11 & 6\\ 5 & 1 & -3\end{bmatrix}, x = -1, y = -2, z = 3\right)$

28. If A = $\begin{pmatrix}3 & -4 & 2\\ 2 & 3 & 5\\ 1 & 0 & 1\end{pmatrix}$, *then find* A^{-1}. *Hence solve the following system of equations* :

3x-4y+2z=-1

2x+3y+5z=7 (DelhiC 2011) $\left(A^{-1} = \frac{1}{-9}\begin{pmatrix}3 & 4 & -26\\ 3 & 1 & -11\\ -3 & -4 & 17\end{pmatrix} x = 3, y = 2 \text{ and } z = -1\right)$

x+z=2

29. If A= $\begin{pmatrix}1 & -2 & 1\\ 0 & -1 & 1\\ 2 & 0 & -3\end{pmatrix}$, *find* A^{-1}. *Hence solve the following system of equations* :

x-2y+z = 0

-y+z = -2 (AIC 2011) $\left(A^{-1} = \frac{1}{1}\begin{pmatrix}3 & -6 & -1\\ 2 & -5 & -1\\ 2 & -4 & -1\end{pmatrix} x = 2, y = 0 \text{ and } z = -2\right)$

2x-3z = 10.

30. If A= $\begin{pmatrix}2 & -1 & 1\\ 3 & 0 & -1\\ 2 & 6 & 0\end{pmatrix}$, *then find* A^{-1}. *Hence solve the following system of equations* :

2x-y+z = -3

3x-z = 0 (AIC 2011) $\left(A^{-1} = \frac{1}{32}\begin{bmatrix}6 & 6 & 1\\ -2 & -2 & 5\\ 18 & -14 & 3\end{bmatrix} x = -\frac{1}{2}, y = \frac{1}{2}, z = -\frac{3}{2}\right)$

2x+6y-2 = 0

31. If A= $\begin{pmatrix}1 & -1 & 2\\ 0 & 2 & -3\\ 3 & -2 & 4\end{pmatrix}$, *then find* A^{-1}. *Hence solve the following system of equations* :

x-y+2z = -7

2y-3z = 13 $\left(A^{-1} = \frac{1}{-1}\begin{pmatrix} 2 & 0 & -1 \\ -9 & -2 & 3 \\ -6 & -1 & 2 \end{pmatrix}; x = 1, y = 2, z = -3\right)$

3x-2y+4z = -13