Sample Question Paper -01 (2018-19)

Mathematics

Class- XII

Time:- 3hrs. M.M.100

General Instructions: **No. Of printed pages: 03**

1. All the questions are compulsory.
2. The question paper consists of 29 questions divided into 4 sections A, B, C and D.
3. Section A comprises of 4 questions of 1 mark each. Section B comprises of 8 questions of 2 marks each. Section C comprises of 11 questions of 4 marks each. Section D comprises of 6 questions of 6 marks each.
4. There is no overall choice. However, an internal choice has been provided in one questions of 1 mark each, three questions of 2 marks each, three questions of 4 marks each and three questions of 6 marks each. You have to attempt only one of the alternatives in all such questions.
5. Use of calculators is not permitted.

Section-A

1. Given a square matrix A of order 3×3, such that $|A| = 10$, find the value of $|\text{adj } A|$. **1**

2. Differentiate x^x with respect to x. **1**

3. What is the degree of the differential equation: $\left(\frac{dy}{dx}\right)^2 - \frac{d^2y}{dx^2} - 6y = \log x$. **1**

4. What are the direction cosines of a line, which makes equal angles with the coordinate axes? **1**

OR

If a line makes angles 90°, 60° and θ with x, y and z-axes respectively. If θ is acute find θ.

Section-B

5. Let A = Q×Q, where Q is the set of all rational numbers and * be a binary operation on A defined by (a, b) * (c, d) = (ac, b+ad) for (a, b), (c, d) ∈ A.. Find the identity element of * in A. **2**

6. If $A = \begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ find k such that $A^2 = 8A + kI$. **2**

7. Evaluate $\int \frac{\sin x}{(1-\cos x)(2-\cos x)} dx$. **2**

8. Evaluate $\int \frac{(2-x)e^x}{(1-x)^2} dx$. **2**

OR

Evaluate $\int \frac{\sqrt{x}}{\sqrt{a^3 - x^3}} dx$.

9. Form the differential equation of the family of circles in the second quadrant and touching the coordinate axes. **2**

10. Find the area of parallelogram having diagonals $3\hat{i} + \hat{j} - 2\hat{k}$ and $\hat{i} - 3\hat{j} + 4\hat{k}$. **2**

OR

Find the projection of $\vec{b} + \vec{c}$ on $\vec{a}$, where $\vec{a} = 2\hat{i} - 2\hat{j} + \hat{k}, \vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + 4\hat{k}$.

11. How many times must a man toss a fair coin so that the probability of having at least one head is more than 80% ? **2**

12. Bag A contains 6 red and 5 blue balls and another bag B contains 5 red and 8 blue balls. One ball from A is transferred from Bag A to B then a ball is drawn from bag B at random. Find the probability that the ball drawn is blue. **2**

OR

Given that P(A) = 0.4, P(B) = 0.7 and P(B/A) = 0.6. Find $P(A\cup B)$.

Section-C

13. Consider $f: R_{+} \to [-9, \infty)$ defined by $f(x) = 5x^2 + 6x - 9$. Prove that f is invertible. Find $f^{-1}(x)$. **4**

OR

Determine whether the relation R defined on the set $\Re$ of all real numbers as R = {(a, b) : a, b ∈ R and $a - b + \sqrt{3} \in S$, where S is the set of all irrational numbers}, is reflexive, symmetric and transitive.

14. Find the value of the expression $\sin\left(2\tan^{-1}\frac{1}{3}\right)+\cos\left(\tan^{-1}2\sqrt{2}\right)$. **4**

15. Using properties of determinants, show that $\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix} = 1+a^2+b^2+c^2$. **4**

16. Rolle's theorem holds on $f(x) = x^3 - 6x^2 + px + q$ in [1, 3] with $c = 2 + \frac{1}{\sqrt{3}}$. Find p and q.

OR

If $f(x) = \begin{cases} \dfrac{1-\cos 4x}{x^2}, & \text{when } x<0 \\ a, & \text{when } x=0 \\ \dfrac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4}, & \text{when } x>0 \end{cases}$ is continuous at x = 0, find the value of a.

17. If $x = a\cos^3\theta$ and $y = a\sin^3\theta$, then find the value of $\frac{d^2y}{dx^2}$ at $\theta = \frac{\pi}{4}$. **4**

18. Find the equations of tangents to the curve $3x^2 - y^2 = 8$, which passes through the point $\left(\frac{4}{3}, 0\right)$ **4**

19. Find $\int \frac{\tan\theta + \tan^3\theta}{1+\tan^3\theta}\, d\theta$. **4**

20. Evaluate $\int_0^{\frac{\pi}{2}} \log(\sin x)\, dx$. **4**

21. Find the particular solution of the differential equation $2x^2\frac{dy}{dx} - 2xy + y^2 = 0$; when y (e) = e. **4**

OR

Find the general solution of the differential equation $(1 + y^2) + (x - e^{\tan^{-1} y})\frac{dy}{dx} = 0$.

22. Show that the vectors $\vec{a}, \vec{b}, \vec{c}$ are coplanar if and only if $\vec{a}+\vec{b}, \vec{b}+\vec{c}$ and $\vec{c}+\vec{a}$ are coplanar. **4**

23. Find the equation of the line passing through the point (1, 2, – 4) and perpendicular to both the lines $\frac{x-8}{3}=\frac{y+19}{-16}=\frac{z-10}{7}$ and $\frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5}$. **4**

Section-D

24. If $A=\begin{bmatrix}2 & -3 & 5\\ 3 & 2 & -4\\ 1 & 1 & -2\end{bmatrix}$, find A^{-1}. Using A^{-1} solve the following system of equations: **6**

$2x - 3y + 5z = 16$; $3x + 2y - 4z = -4$; $x + y - 2z = -3$.

OR

Using the elementary operations find the inverse of matrix $A=\begin{bmatrix}3 & 0 & -1\\ 2 & 3 & 0\\ 0 & 4 & 1\end{bmatrix}$

25. Given the sum of the perimeters of a square and a circle, show that the sum of their areas is least when the side of the square is equal to the diameter of the circle. **6**

26. Using integration find the area of the region in the first quadrant enclosed by the x-axis, the line $x = \sqrt{3}\,y$ and the circle $x^2 + y^2 = 4$. **6**

OR

Find the area of the region $\{(x, y) : y^2 \le 6ax \text{ and } x^2 + y^2 \le 16a^2\}$ using method of integration.

27. Find the vector equation of the plane passing through the points (2, 1, – 1) and (– 1, 3, 4) and perpendicular to the plane $x - 2y + 4z = 10$. Also show that the plane thus obtained contains the line $\vec{r} = -\hat{i} + 3\hat{j} + 4\hat{k} + \lambda(3\hat{i} - 2\hat{j} - 5\hat{k})$. **6**

OR

Show that the lines $\frac{x+3}{-3}=\frac{y-1}{1}=\frac{z-5}{5}$ and $\frac{x+1}{-1}=\frac{y-2}{2}=\frac{z-5}{5}$ are coplanar. Also find the equation of the plane containing the lines.

28. A toy company manufactures two types of dolls, A and B. Market tests and available resources have indicated that the combined production level should not exceed 1200 dolls per week and the demand for dolls of type B is at most half of that for dolls of type A. Further, the production level of the dolls of type A can exceed three times the production of dolls of other type at most 600 units. If the company makes profit of ₹12 and ₹16 per doll respectively on dolls A and B, how many dolls of each type should be produced weekly in order to maximize profit ? Formulate LPP and solve it graphically. **6**

29. Bag I contains 3 red and 4 black balls and bag II 4 red and 5 black balls. One ball is transferred from bag I to bag II and then a ball is drawn from bag II at random. The ball so drawn is found to be red in colour. Find the probability that the transferred ball is black. **6**

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Mathematics

Class- XII

Q	Section-A	Marks
1.	100.	1
2.	$x^x (1 + \log x)$	1
3.	Degree = 1	1
4.	$\left\langle \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}} \right\rangle$ ***OR*** 30°	1
	Section-B	
5.	Let $(e_1, e_2) \in A$ is identity element, then $(a, b) * (e_1, e_2)\ (a, b) = (e_1, e_2) * (a, b)$ $\Rightarrow (ae_1, b+ae_2) = (a, b) \quad \Rightarrow e_1 = 1$ and $e_2 = 0$. $\quad \therefore$ Identity element is (1, 0).	½ 1+½
6.	$\begin{bmatrix} 1 & 0 \\ -8 & 49 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ -8 & 56 \end{bmatrix} + \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 \\ -8 & 49 \end{bmatrix} = \begin{bmatrix} 8+k & 0 \\ -8 & 56+k \end{bmatrix} \Rightarrow k = -7$	1+½ +½
7.	Let $\cos x = t$, then $-\sin x\,dx = dt$. $= \int \frac{-dt}{(1-t)(2-t)} = -\int \frac{(2-t)-(1-t)}{(1-t)(2-t)} dt = -\int \frac{dt}{1-t} + \int \frac{dt}{2-t} = \log\|1-t\| - \log\|2-t\| + C = \log\left\|\frac{1-\cos x}{2-\cos x}\right\| + C$	½ 1+½
8.	$\int \frac{(1+1-x)e^x}{(1-x)^2} dx = \int e^x \left\{\frac{1}{(1-x)^2} + \frac{1}{1-x}\right\} dx = \frac{1}{1-x} \cdot e^x + C$, using $\int e^x \{f(x) + f'(x)\} dx = f(x).e^x + C$ ***OR*** Let $x^{3/2} = t \Rightarrow \frac{3}{2}\sqrt{x}\,dx = dt \Rightarrow \sqrt{x}\,dx = \frac{2}{3}dt$ $\frac{2}{3}\int \frac{1}{\sqrt{(a^{3/2})^2 - t^2}} dt = \frac{2}{3}\sin^{-1}\frac{t}{a^{3/2}} + C = \frac{2}{3}\sin^{-1}\frac{x^{3/2}}{a^{3/2}} + C = \frac{2}{3}\sin^{-1}\sqrt{\frac{x^3}{a^3}} + C$	½+1 +½ ½ 1½
9.	Let radius of circle be a, then centre in second quadrant is (– a, a) as circle touches coordinate axes. Equation of circle is $(x + a)^2 + (y - a)^2 = a^2$ or $x^2 + y^2 + 2a(x - y) - a^2 = 0$ … (i) Diff w.r.t. x, we get $2x + 2yy' + 2a(1 - y') - 0 = 0 \Rightarrow a = -\frac{x + yy'}{1 - y'}$ put in (i) $x^2 + y^2 + 2\left(-\frac{x+yy'}{1-y'}\right)(x - y) - \left(-\frac{x+yy'}{1-y'}\right)^2 = 0 \quad \Rightarrow (x^2 + 2xy)y'^2 - 2xyy' + y^2 + 2xy = 0$	½ ½ ½ ½

10.	Let $\vec{d}_1 = 3\hat{i}+\hat{j}-2\hat{k}$ and $\vec{d}_2 = \hat{i}-3\hat{j}+4\hat{k}$. then $\vec{d}_1 \times \vec{d}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & -1 \\ 1 & -3 & 4 \end{vmatrix} = -2\hat{i}-14\hat{j}-10\hat{k}$	1
	Area of parallelogram = $\frac{1}{2}\left\|\vec{d}_1 \times \vec{d}_2\right\| = \frac{1}{2}\left\|-2\hat{i}-14\hat{j}-10\hat{k}\right\| = \frac{1}{2}\sqrt{4+196+100} = 5\sqrt{3}$ sq units.	1
	OR	
	$\vec{b}+\vec{c} = 3\hat{i}+\hat{j}+2\hat{k}$	½
	Projection of $\vec{b}+\vec{c}$ on $\vec{a} = \frac{(\vec{b}+\vec{c})\cdot\vec{a}}{\|\vec{a}\|} = \frac{(3\hat{i}+\hat{j}+2\hat{k})\cdot(2\hat{i}-2\hat{j}+\hat{k})}{\sqrt{4+4+1}} = \frac{6-2+2}{3} = \frac{6}{3} = 2.$	1½
11.	Let a coin be tossed n times. p = prob of getting a head = ½, q = 1 – ½ = ½	½
	$P(r) = {}^nC_r\, q^{n-r} p^r = {}^nC_r \left(\frac{1}{2}\right)^{n-r}\left(\frac{1}{2}\right)^r = {}^nC_r\left(\frac{1}{2}\right)^n$	½
	P(at least one head) > 80% i.e. 1 – P(0) > 80/100 or 1 – 4/5 > P(0) or P(0) < 1/5	
	$\left(\frac{1}{2}\right)^n < \frac{1}{5} \Rightarrow n = 3$, as $\frac{1}{8} < \frac{1}{5}$.	
	Therefore a coin must be tossed three times or more than three times.	1
12.	Case I: If a red ball is transferred from A, probability of drawing blue ball from B, $P_1 = \frac{6}{11} \times \frac{8}{14} = \frac{48}{154}$	½
	Case II: If a red ball is transferred from A, probability of drawing blue ball from B, $P_2 = \frac{5}{11} \times \frac{9}{14} = \frac{45}{154}$	½
	Therefore required probability = $P_1 + P_2 = \frac{48}{154} + \frac{45}{154} = \frac{93}{154}$	1
	OR	
	$P(A \cap B) = P(A) \times P(B/A) = 0.4 \times 0.6 = 0.24$	1
	$P(A \cap B) = P(A) \times P(B/A) = 0.4 \times 0.6 = 0.24$	1
	Section-C	
13.	One-One: Let for $x_1, x_2 \in R_+$, $f(x_1) = f(x_2) \Rightarrow (x_1 - x_2)\,[5(x_1 + x_2) + 6] = 0$	
	$\Rightarrow x_1 - x_2 = 0$ or $x_1 + x_2 = -6/5$ (reject) $\Rightarrow x_1 = x_2$. Hence one-one.	1
	Onto: Let $y \in [-9, \infty)$, $\exists\, x \in R_+$, s.t. $y = f(x) \Rightarrow y = 5x^2 + 6x - 9$ or $5x^2 + 6x - 9 - y = 0$	
	$\Rightarrow x = \frac{-6 \pm \sqrt{36 - 20(-9-y)}}{10} = \frac{\sqrt{5y+54}-3}{5} \in R_+$. f is onto. Hence f is invertible.	2
	$\Rightarrow y = f(x)$ $\Rightarrow x = f^{-1}(y) = \frac{\sqrt{5y+54}-3}{5} \Rightarrow f^{-1}(x) = \frac{\sqrt{5x+54}-3}{5}$	1

OR

Reflexive: $\forall\, a \in \Re$, $(a, a) \in R \Rightarrow a - a + \sqrt{3} \in S \quad \Rightarrow \sqrt{3} \in S$, true. Hence reflexive. — 1

Symmetric: For $a, b \in \Re$, $(a, b) \in R \Rightarrow a - b + \sqrt{3} \in S \Rightarrow b - a + \sqrt{3} \in S \Rightarrow (b, a) \in R$. — 1

Transitive: Let $a = 2 + \sqrt{3}$, $b = 5$, $c = 4 + 3\sqrt{3}$, clearly $(a, b) \in R$, i.e. $-2 + \sqrt{3} \in S$. Also, $(b, c) \in R$, i.e. $1 - 2\sqrt{3} \in S$ but $(a, c) \notin R$ as $-1 \notin S$. Hence not transitive. — 2

14.

$$= \sin\left(\tan^{-1}\frac{3}{4}\right) + \cos\left(\tan^{-1} 2\sqrt{2}\right) \left(\text{using } 2\tan^{-1} x = \tan^{-1}\frac{2x}{1-x^2}\right)$$ — 1

$$= \sin\left(\cot^{-1}\frac{4}{3}\right) + \cos\left(\tan^{-1} 2\sqrt{2}\right) = \frac{1}{\operatorname{cosec}\left(\cot^{-1}\frac{4}{3}\right)} + \frac{1}{\sec\left(\tan^{-1} 2\sqrt{2}\right)} =$$ — 1

$$\frac{1}{\sqrt{1+\cot^2\left(\cot^{-1}\frac{4}{3}\right)}} + \frac{1}{\sqrt{1+\tan^2\left(\tan^{-1} 2\sqrt{2}\right)}} = \frac{1}{\sqrt{1+\frac{16}{9}}} + \frac{1}{\sqrt{1+8}} = \frac{3}{5} + \frac{1}{3} = \frac{14}{15}$$ — 2

15.

$$= \frac{1}{abc}\begin{vmatrix} a(a^2+1) & a^2 b & a^2 c \\ ab^2 & b(b^2+1) & b^2 c \\ c^2 a & c^2 b & c(c^2+1) \end{vmatrix}$$ (On multiplying R_1, R_2 and R_3 by a, b and c respectively) — ½

$$= \frac{abc}{abc}\begin{vmatrix} a^2+1 & a^2 & a^2 \\ b^2 & b^2+1 & b^2 \\ c^2 & c^2 & c^2+1 \end{vmatrix}$$ (taking common a, b and c from C_1, C_2 and C_3 by respectively) — ½

$$= \begin{vmatrix} 1+a^2+b^2+c^2 & 1+a^2+b^2+c^2 & 1+a^2+b^2+c^2 \\ b^2 & b^2+1 & b^2 \\ c^2 & c^2 & c^2+1 \end{vmatrix}$$ $(R_1 \to R_1 + R_2 + R_3)$ — 1

$$= (1+a^2+b^2+c^2)\begin{vmatrix} 1 & 1 & 1 \\ b^2 & b^2+1 & b^2 \\ c^2 & c^2 & c^2+1 \end{vmatrix} = (1+a^2+b^2+c^2)\begin{vmatrix} 1 & 0 & 0 \\ b^2 & 1 & 0 \\ c^2 & 0 & 1 \end{vmatrix}$$ $(C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1)$ — 1

$= (1+a^2+b^2+c^2)\{1(1-0) - 0 + 0\} = 1+a^2+b^2+c^2$ (expanding along R_1) — 1

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16. $f(1) = f(3) \Rightarrow 1 - 6 + p + q = 27 - 54 + 3p + q \Rightarrow p = 11$ — 2

$f'(x) = 3x^2 - 12x + p$. But $f'\left(2 + \frac{1}{\sqrt{3}}\right) = 0 \Rightarrow 3\left(2 + \frac{1}{\sqrt{3}}\right)^2 - 12\left(2 + \frac{1}{\sqrt{3}}\right) + p = 0 \Rightarrow$ q can be any number — 2

OR

For continuity at $x = 0$, we have LHL = RHL = f(0) — ½

$$\text{LHL} = \lim_{x \to 0^-} f(x) = \lim_{x \to 0} \frac{1-\cos 4x}{x^2} = \lim_{x \to 0} \frac{2\sin^2 2x}{x^2} = \lim_{x \to 0} 8\left(\frac{\sin^2 2x}{2x}\right) = 8 \times 1^2 = 8$$ — 1½

$$\text{RHL} = \lim_{x \to 0^+} f(x) = \lim_{x \to 0} \frac{\sqrt{x}}{\sqrt{16+\sqrt{x}} - 4} = \lim_{x \to 0} \frac{\sqrt{x}\left(\sqrt{16+\sqrt{x}} + 4\right)}{16 + \sqrt{x} - 16} = \lim_{x \to 0}\left(\sqrt{16+\sqrt{x}} + 4\right) = 4 + 4 = 8$$ — 1½

Therefore, $f(0) = a = 8$ — ½

17.	$\frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta$ and $\frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta \Rightarrow \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = -\tan\theta$ $\therefore \frac{d^2y}{dx^2} = -\sec^2\theta \times \frac{d\theta}{dx} = \frac{-\sec^2\theta}{-3a\cos^2\theta\sin\theta} = \frac{1}{3a}\sec^4\theta\,\mathrm{cosec}\theta \Rightarrow \left.\frac{d^2y}{dx^2}\right\|_{\theta=\frac{\pi}{4}} = \frac{4\sqrt{2}}{3a}$	2½ 1½
18.	Let the tangent to the curve $3x^2 - y^2 = 8$ pass through (4/3, 0) at (x_1, y_1). (x_1, y_1) lies on the curve $3x^2 - y^2 = 8$, therefore $3{x_1}^2 - {y_1}^2 = 8$ … (i) Differentiating $3x^2 - y^2 = 8$ w.r.t. x, we get $6x - 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{3x}{y} \Rightarrow \left.\frac{dy}{dx}\right\|_{(x_1,y_1)} = \frac{3x_1}{y_1}$ Equation of tangent is $y - y_1 = \frac{3x_1}{y_1}(x - x_1)$ If tangent pass through (4/3, 0) then $0 - y_1 = \frac{3x_1}{y_1}\left(\frac{4}{3} - x_1\right) \Rightarrow 3{x_1}^2 - {y_1}^2 = 4x_1 \Rightarrow 8 = 4x_1 \Rightarrow x_1 = 2$ (from (i)) $12 - {y_1}^2 = 8 \Rightarrow {y_1}^2 = 4 \Rightarrow y_1 = \pm 2$. Required point is $(2, \pm 2)$ and equation of tangents are $3x - y - 4 = 0$ and $3x + y - 4 = 0$.	½ 1 ½ 1 1
19.	$\int \frac{\tan\theta + \tan^3\theta}{1+\tan^3\theta}d\theta = \int \frac{\tan\theta(1+\tan^2\theta)}{1+\tan^3\theta}d\theta = \int \frac{\tan\theta(\sec^2\theta)}{1+\tan^3\theta}d\theta = \int \frac{t}{1+t^3}dt$ (Let $\tan\theta = t \Rightarrow \sec^2\theta\, d\theta = dt$) Let $\frac{t}{1+t^3} = \frac{t}{(1+t)(1-t+t^2)} = \frac{A}{1+t} + \frac{Bt+C}{1-t+t^2} \Rightarrow A = -\frac{1}{3}, B = \frac{1}{3}, C = \frac{1}{3}$ $\int \frac{t}{1+t^3}dt = -\frac{1}{3}\int \frac{dt}{1+t} + \frac{1}{3}\int \frac{t+1}{1-t+t^2}dt = -\frac{1}{3}\int \frac{dt}{1+t} + \frac{1}{6}\int \frac{(2t-1)+3}{1-t+t^2}dt$ $= -\frac{1}{3}\int \frac{dt}{1+t} + \frac{1}{6}\int \frac{(2t-1)}{1-t+t^2}dt + \frac{1}{2}\int \frac{1}{\left(t-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}dt$ $= -\frac{1}{3}\log\|1+t\| + \frac{1}{6}\log\|t^2 - t + 1\| + \frac{1}{2}\cdot\frac{2}{\sqrt{3}}\tan^{-1}\frac{(t-1/2)}{\sqrt{3}/2} + C =$ $= -\frac{1}{3}\log\|1+\tan\theta\| + \frac{1}{6}\log\|\tan^2\theta - \tan\theta + 1\| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan\theta - 1}{\sqrt{3}}\right) + C$	1 1 1 1
20.	Let $I = \int_0^{\frac{\pi}{2}} \log(\sin x)dx \Rightarrow I = \int_0^{\frac{\pi}{2}} \log\{\sin(\pi/2 - x)\}dx = \int_0^{\frac{\pi}{2}} \log(\cos x)dx$ by $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ Add, $2I = \int_0^{\frac{\pi}{2}} \log(\sin x\cos x)dx = \int_0^{\frac{\pi}{2}} \log\left(\frac{\sin 2x}{2}\right)dx = \int_0^{\frac{\pi}{2}} [\log(\sin 2x) - \log 2]dx = I_1 - \log 2[x]_0^{\frac{\pi}{2}} = I_1 - \frac{\pi}{2}\log 2$ $I_1 = \int_0^{\frac{\pi}{2}} \log(\sin 2x)dx = \frac{1}{2}\int_0^{\pi} \log(\sin t)dt$ (Let $2x = t \Rightarrow 2dx = dt$ if $x = 0 \Rightarrow t = 0$ and if $x = \frac{\pi}{2} \Rightarrow t = \pi$) By property $\int_0^{2a} f(x)dx = 2\int_0^a f(2a-x)dx$, if $f(2a-x) = f(x)$, we get $I_1 = \frac{1}{2}\cdot 2\int_0^{\frac{\pi}{2}} \log(\sin t)dt = \int_0^{\frac{\pi}{2}} \log(\sin x)dx = I$ $2I = I - \frac{\pi}{2}\log 2 \qquad \Rightarrow I = -\frac{\pi}{2}\log 2$	1 1 ½ 1 ½

21. $\frac{dy}{dx}=\frac{2xy-y^2}{2x^2}$ is homogeneous differential equation. Let $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ 1

$\therefore\ v + x\frac{dv}{dx} = \frac{2x^2v - x^2v^2}{2x^2} = \frac{2v - v^2}{2} \Rightarrow x\frac{dv}{dx} = \frac{2v - v^2}{2} - v = \frac{2v - v^2 - 2v}{2} = -\frac{v^2}{2} \Rightarrow \frac{dv}{v^2} = -\frac{dx}{2x}$ 1

$\int\frac{dv}{v^2} = -\frac{1}{2}\int\frac{dx}{x} \Rightarrow -\frac{1}{v} = -\frac{1}{2}\log|x| + C \Rightarrow -\frac{x}{y} = -\frac{1}{2}\log|x| + C$ 1

Given $y = e$, when $x = e$, $\Rightarrow -1 = -\frac{1}{2}\log e + C \Rightarrow C = -\frac{1}{2}$

Particular solution is $-\frac{x}{y} = -\frac{1}{2}\log|x| - \frac{1}{2} \Rightarrow \frac{2x}{y} = \log|x| + 1$ 1

OR

$\frac{dy}{dx} = \frac{-(1+y^2)}{x - e^{\tan^{-1}y}} \Rightarrow \frac{dx}{dy} = \frac{e^{\tan^{-1}y} - x}{1+y^2} \Rightarrow \frac{dx}{dy} + \frac{x}{1+y^2} = \frac{e^{\tan^{-1}y}}{1+y^2}$ it is LDE of the form $\frac{dx}{dy} + P(y)\cdot x = Q(y)$ 1

I.F. $= e^{\int\frac{1}{1+y^2}dy} = e^{\tan^{-1}y}$ and solution is $(\text{I.F.})x = \int(\text{I.F.})\cdot Q(y)dy$ 1

i.e. $e^{\tan^{-1}y}\cdot x = \int e^{\tan^{-1}y}\cdot\frac{e^{\tan^{-1}y}}{1+y^2}dy = \int\frac{e^{2\tan^{-1}y}}{1+y^2}dy = \int e^{2t}dt = \frac{e^{2t}}{2} + C$ (Let $\tan^{-1}y = t \Rightarrow \frac{1}{1+y^2}dy = dt$) 1

i.e. $e^{\tan^{-1}y}\cdot x = \frac{1}{2}e^{2\tan^{-1}y} + C \qquad \Rightarrow x = \frac{1}{2}e^{\tan^{-1}y} + Ce^{-\tan^{-1}y}$ 1

22. Consider the scalar triple product $[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}] = (\vec{a}+\vec{b})\cdot\{(\vec{b}+\vec{c})\times(\vec{c}+\vec{a})\} = 2[\vec{a}\ \vec{b}\ \vec{c}]$ 2

Let $\vec{a}+\vec{b}, \vec{b}+\vec{c}$ and $\vec{c}+\vec{a}$ are coplanar $[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}] = 0 \Rightarrow 2[\vec{a}\ \vec{b}\ \vec{c}] = 0 \Rightarrow [\vec{a}\ \vec{b}\ \vec{c}] = 0$

$\Rightarrow \vec{a}, \vec{b}, \vec{c}$ are coplanar. 1

Conversely let $\vec{a}, \vec{b}, \vec{c}$ are coplanar $\Rightarrow [\vec{a}\ \vec{b}\ \vec{c}] = 0 \Rightarrow 2[\vec{a}\ \vec{b}\ \vec{c}] = 0 \Rightarrow [\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}] = 0$

$\Rightarrow \vec{a}+\vec{b}, \vec{b}+\vec{c}$ and $\vec{c}+\vec{a}$ are coplanar. 1

Hence $\vec{a}, \vec{b}, \vec{c}$ are coplanar if and only if $\vec{a}+\vec{b}, \vec{b}+\vec{c}$ and $\vec{c}+\vec{a}$ are coplanar.

23. Equation of a line through (1, 2, – 4) is $\frac{x-1}{a} = \frac{y-2}{b} = \frac{z+4}{c}$ 1

It is perpendicular to both the lines $\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7}$ and $\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}$

Then $3a - 16b + 7c = 0$ and $3a + 8b - 5c = 0$ 1

$\Rightarrow \frac{a}{80-56} = \frac{b}{-15-21} = \frac{c}{24+48} \Rightarrow \frac{a}{24} = \frac{b}{36} = \frac{c}{72} \Rightarrow \frac{a}{2} = \frac{b}{3} = \frac{c}{6}$ so direction ratios are 2, 3, 6 1

And equation of line is $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z+4}{6}$ 1

Section-D

24. $|A| = \begin{vmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{vmatrix} = 2(0) + 3(-2) + 5(1) = -1 \neq 0$ $\text{adj}\,A = \begin{bmatrix} 0 & 2 & 1 \\ -1 & -9 & -5 \\ 2 & 23 & 13 \end{bmatrix}^T = \begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}$ 1+2

$$A^{-1} = \frac{1}{|A|}\text{adj}\,A = \frac{1}{-1}\begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}$$ 1

For given equations the matrix equation is AX = B. Its solution is $X = A^{-1}B$. i.e.

$$\begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 16 \\ -4 \\ -3 \end{bmatrix} \quad \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}\begin{bmatrix} 16 \\ -4 \\ -3 \end{bmatrix} = \begin{bmatrix} 0-4+6 \\ -32-36+69 \\ -16-20+39 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$ 1½

Therefore solution is x = 2, y = 1 and z = 3.

½

OR

$$\text{Let } A = IA \Rightarrow \begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 0 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}A \quad \Rightarrow \begin{bmatrix} 1 & -3 & -1 \\ 2 & 3 & 0 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}A \quad (R_1 \to R_1 - R_2)$$ ½+1

$$\Rightarrow \begin{bmatrix} 1 & -3 & -1 \\ 0 & 9 & 2 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}A \quad (R_2 \to R_2 - 2R_1) \quad \Rightarrow \begin{bmatrix} 1 & -3 & -1 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -2 \\ 0 & 0 & 1 \end{bmatrix}A \quad (R_2 \to R_2 - 2R_3)$$ 1+1

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -5 & 8 & -6 \\ -2 & 3 & -2 \\ 8 & -12 & 9 \end{bmatrix}A \quad \begin{pmatrix} R_1 \to R_1 + 3R_2 \\ R_3 \to R_3 - 4R_2 \end{pmatrix} \quad \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -4 & 3 \\ -2 & 3 & -2 \\ 8 & -12 & 9 \end{bmatrix}A \quad (R_1 \to R_1 + R_3)$$ 1+½

$$\text{Hence, } A^{-1} = \begin{bmatrix} 3 & -4 & 3 \\ -2 & 3 & -2 \\ 8 & -12 & 9 \end{bmatrix}$$ 1

25. Let the side of a square be x and radius of a circle be r. Sum of their perimeters $P = 4x + 2\pi r$ 1

Sum of their areas $A = x^2 + \pi r^2 = \left(\frac{P - 2\pi r}{4}\right)^2 + \pi r^2$. 1½

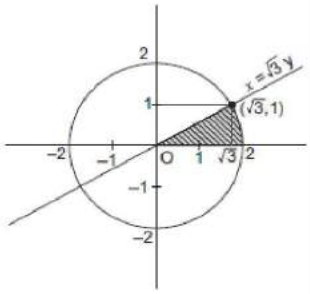
$$\frac{dA}{dr} = \frac{1}{16}\cdot 2(P - 2\pi r)(-2\pi) + 2\pi r = -\frac{\pi}{4}(P - 2\pi r) + 2\pi r$$

For minimum area, $\frac{dA}{dr} = 0 \Rightarrow \frac{\pi}{4}(P - 2\pi r) = 2\pi r \Rightarrow P - 2\pi r = 8r \Rightarrow x = 2r$ 2

$$\frac{d^2A}{dr^2} = -\frac{\pi}{4}(-2\pi) + 2\pi = \left(\frac{\pi^2}{2} + 2\pi\right) \Rightarrow \left.\frac{d^2A}{dr^2}\right]_{x=2r} > 0$$ 1

Hence, the sum of areas is least when the side of the square is equal to the diameter of the circle. ½

26. Given circle $x^2 + y^2 = 4$ and line $x = \sqrt{3}\,y$. Solving the equations their point of intersection in the first quadrant is $(\sqrt{3}, 1)$. — Fig-1, 1



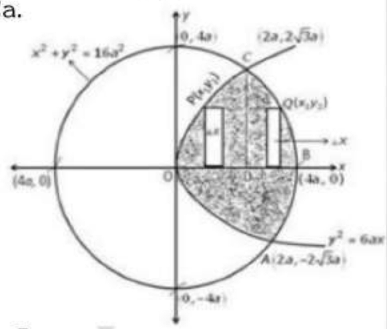
Required shaded area $= \int_0^{\sqrt{3}} y_{\text{line}}\,dx + \int_{\sqrt{3}}^{2} y_{\text{circle}}\,dx = \int_0^{\sqrt{3}} \frac{x}{\sqrt{3}}\,dx + \int_{\sqrt{3}}^{2} \sqrt{4-x^2}\,dx$ — 1

$$= \left[\frac{x^2}{2\sqrt{3}}\right]_0^{\sqrt{3}} + \left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_{\sqrt{3}}^{2}$$ — 2

$$= \frac{3}{2\sqrt{3}} - 0 + (0 + 2\sin^{-1}1) - \left(\frac{\sqrt{3}}{2}\sqrt{4-3} + 2\sin^{-1}\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{2} + 2\frac{\pi}{2} - \frac{\sqrt{3}}{2} - 2\frac{\pi}{3} = \pi - \frac{2\pi}{3} = \frac{\pi}{3} \text{ sq units}$$ — 1

OR

Equations of corresponding inequations are $y^2 = 6ax$ and $x^2 + y^2 = 16a^2$. Eliminating y from them we get $x^2 + 6ax - 16a^2 = 0 \Rightarrow (x + 8a)(x - 2a) = 0 \Rightarrow x = -8a$ (Reject), 2a. — Fig.-1, 1



Requiredshaded area $= \int_0^{2a} y_{\text{parabola}}\,dx + \int_{2a}^{4a} y_{\text{circle}}\,dx$ — 1

$$= \int_0^{2a} \sqrt{6ax}\,dx + \int_{2a}^{4a} \sqrt{16a^2 - x^2}\,dx$$

$$= 2\left[\sqrt{6}\sqrt{a}\,\frac{x\sqrt{x}}{3}\right]_0^{2a} + 2\left[\frac{x}{2}\sqrt{16a^2 - x^2} + \frac{16a^2}{2}\sin^{-1}\left(\frac{x}{4a}\right)\right]_{2a}^{4a}$$

$$= 2\left[\frac{2}{3}\sqrt{6}\sqrt{a}(2a)^{3/2} - 0\right] + 2\left[\{0 + 8a^2\sin^{-1}1\} - \frac{2a}{2}\sqrt{16a^2 - 4a^2} + 8a^2\sin^{-1}\left(\frac{1}{2}\right)\right]$$ — 2

$$= 2\left[\frac{8\sqrt{3}}{3}a^2 + 4a^2\pi - 2\sqrt{3}a^2 - \frac{4a^2\pi}{3}\right] = 2\left[\frac{2\sqrt{3}}{3}a^2 + \frac{8\pi a^2}{3}\right] = \frac{4a^2}{3}(\sqrt{3} + 4\pi) \text{ sq units}$$ — 1

27. General equation of a plane through (2, 1, – 1) is $a(x - 2) + b(y - 1) + c(z + 1) = 0$ .... (i) — 1

Plane (i) is passes through (– 1, 3, 4) $\Rightarrow$ $-3a + 2b + 5c = 0$ .... (ii) — 1

Plane (i) is perpendicular to $x - 2y + 4z = 10 \Rightarrow$ $a - 2b + 5c = 0$ .... (iii) — 1

From (i), (ii), (iii) eliminating a, b, c, we get $\begin{vmatrix} x-2 & y-1 & z+1 \\ -3 & 2 & 5 \\ 1 & -2 & 4 \end{vmatrix} = 0 \quad \Rightarrow 18x + 17y + 4z - 49 = 0$ — 1

Equation of plane in vector form is $\vec{r}\cdot(18\hat{i} + 17\hat{j} + 4\hat{k}) - 49 = 0$

If planecontains $\vec{r} = -\hat{i} + 3\hat{j} + 4\hat{k} + \lambda(3\hat{i} - 2\hat{j} - 5\hat{k})$, then $(-\hat{i} + 3\hat{j} + 4\hat{k})\cdot(18\hat{i} + 17\hat{j} + 4\hat{k}) = 0$ — 1

i.e. $-18 + 51 + 16 - 49 = 0$, this is true. And $(3\hat{i} - 2\hat{j} - 5\hat{k})\cdot(18\hat{i} + 17\hat{j} + 4\hat{k}) = 0 \Rightarrow 54 - 34 - 20 = 0$, true. — 1

Hence line lies in the plane.

OR

If lines are coplanar, then $\begin{vmatrix} -3+1 & 1-2 & 5-5 \\ -3 & 1 & 5 \\ -1 & 2 & 5 \end{vmatrix} = -2(5-10)+1(-15+5)+0 = 10-10 = 0$, true.

4

Equation of the plane containing the lines is

$$\begin{vmatrix} x+3 & y-1 & z-5 \\ -3 & 1 & 5 \\ -1 & 2 & 5 \end{vmatrix} = 0 \Rightarrow (x+3)(-5)-(y-1)(-10)+(z-5)(-5)=0$$
$$\Rightarrow x-2y+z=0$$

2

28. Let x and y be the number of dolls of type A and B respectively.

Problem can be formulated as

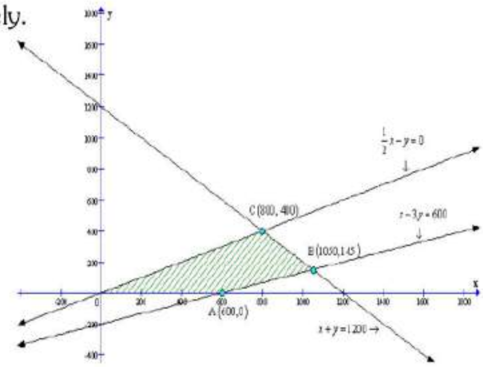
Maximise $Z = 12x + 16y$. Subject to constraints,

1

$x + y \le 1200$, $x \ge 2y$, $x - 3y \le 600$ and $x, y \ge 0$.

1

From the graph, shaded part is the feasible region.



Corner Points	Z= 12x + 16y	
A (600, 0)	7200	
B (1050, 150)	15000	
C (800, 400)	16000	→ Maximum

Table-1

Graph -2

The maximum value of Z is ₹16000 at C (800, 400). Thus 800 and 400 dolls of types A and B should be produced to get the maximum profit.

1

29. Bag I: 3 red + 4 black, Bag II: 4 red + 5 black. Let E_1: Black ball is transferred from bag I

E_2: Red ball is transferred from bag I, A: Black ball is drawn from bag II

1

Then $P(E_1) = 4/7$ and $P(A/E_1) = 4/10$ and $P(E_2) = 3/7$ and $P(A/E_2) = 5/10$

2

Using Bayes' Theorem $P(E_1/A) = \dfrac{P(E_1)\cdot P(A/E_1)}{P(E_1)\cdot P(A/E_1) + P(E_2)\cdot P(A/E_2)}$

1

$$P(E_1/A) = \frac{\frac{4}{7}\times\frac{4}{10}}{\frac{4}{7}\times\frac{4}{10}+\frac{3}{7}\times\frac{5}{10}} = \frac{16}{16+15} = \frac{16}{31}.$$

2

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